DPG Fundamentals

UW formulation

$$\begin{cases} u \in D(A) \\ Au = f \end{cases} \Rightarrow \begin{cases} u \in L^2(\Omega) \\ (u, A^* v) = (f, v) \; v \in D(A^*) \end{cases} \Rightarrow \begin{cases} u \in L^2(\Omega),\ \hat{u} \in \hat{U} \\ (u, A^* v) + \langle \hat{u}, v \rangle = (f, v) \; v \in H_{A^*}(\Omega_h) \end{cases}$$

Inf--sup constant γ depends upon boundedness below constant α and scaling parameter β in the adjoint graph norm

$$\left. \begin{array}{l} \alpha \|u\| \le \|Au\|,\ u \in D(A) \\ \|v\|_V^2 := \|A^* v\|^2 + \beta^2 \|v\|^2 \end{array} \right\} \quad \Rightarrow \quad \gamma \ge [1 + (\frac{\beta}{\alpha})^2]^{-1/2} .$$

(Ideal) DPG reproduces the stability of the continuous problem

$$\underbrace{\|u - u_h\|^2}_{L^2-\text{error}} \le \underbrace{[1 + (\frac{\beta}{\alpha})^2]}_{\text{stability constant}} \{ \underbrace{\inf_{w_h \in U_h} \|u - w_h\|^2}_{\text{field BA error}} + \underbrace{\inf_{\hat{w}_h \in \hat{U}_h} \|\hat{u} - \hat{w}_h\|^2}_{\text{trace BA error}} \}$$

Full Envelope UW Formulation for Linear Waveguide Problem (1)

Def. Full envelope operator

$$\widetilde{A}\widetilde{u} := e^{ikz} A(e^{-ikz}\widetilde{u})$$

Thm. Full envelope operator inherits boundedness below constant from the original operator

$$\|Au\| \geq \alpha\|u\| \quad \Leftrightarrow \quad \|\widetilde{A}\widetilde{u}\| \geq \alpha\|\widetilde{u}\|$$

Proof:

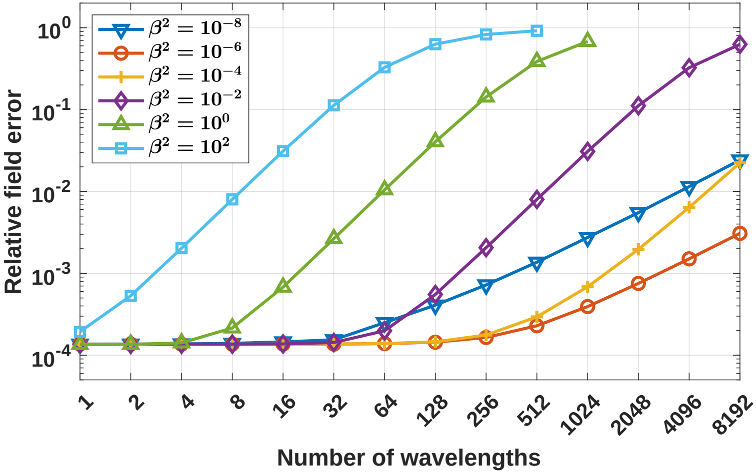
$$\|\widetilde{A}\widetilde{u}\| = \|e^{ikz} A(e^{-ikz}\widetilde{u})\| = \|A(e^{-ikz}\widetilde{u})\| \geq \alpha\|e^{-ikz}\widetilde{u}\| = \alpha\|\widetilde{u}\|$$

Thm. The boundedness below constant depends inversely linearly upon waveguide length L (the subject of this talk)

$$\|Au\| \geq \underbrace{\frac{\alpha_0}{L}}_{=:\alpha} \|u\|$$

[1] M. Melenk, L. Demkowicz, and S. Henneking, ``Stability analysis for electromagnetic waveguides. Part 1: Acoustic and homogeneous electromagnetic waveguides.,'' Oden Institute, The University of Texas at Austin, Austin, TX 78712, Tech. Rep. 2, 2023

Positive Effect of Small β on Pollution



Pollution error in a 3D rectangular waveguide for ultraweak DPG Maxwell with test norm:

$$\|v\|^2_{V(\Omega_h)} = \|\operatorname{curl} F - i\omega\epsilon G\|^2 + \|\operatorname{curl} G + i\omega F\|^2 + \beta^2 \left(\|F\|^2 + \|G\|^2\right).$$

ANALYSIS OF A NON-HOMOGENEOUS EM WAVEGUIDE PROBLEM(2)

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2 L. Demkowicz, M. Melenk, S. Henneking, and J. Badger, ``Stability analysis for acoustic and electromagnetic waveguides. Part 2: Non-homogeneous waveguides.,'' Oden Institute, The University of Texas at Austin, Austin, TX 78712, Tech. Rep. 3, 2023

Outline

Eigensystems

Let $E = (\underbrace{E_1, E_2}_{=E_t}, E_3)$, $e_z = (0, 0, 1)$. We will use the 2D identities:

$$\begin{aligned} e_z \times (e_z \times E_t) &= -E_t \\ e_z \times (\nabla \times E_3) &= \nabla E_3 & \qquad e_z \times \nabla E_3 &= -\nabla \times E_3 \\ \operatorname{curl}(e_z \times E_t) &= \operatorname{div} E_t & \qquad \operatorname{div}(e_z \times E_t) &= -\operatorname{curl} E_t \,. \end{aligned}$$

The original system of equations,

$$\nabla \times E - i\omega H = f \qquad \nabla \times H + i\omega\epsilon E = g$$

translates into:

$$\left\{ \begin{aligned} \nabla \times E_3 + e_z \times \frac{\partial}{\partial z} E_t - i\omega H_t &= f_t \\ \operatorname{curl} E_t - i\omega H_3 &= f_3 \\ \nabla \times H_3 + e_z \times \frac{\partial}{\partial z} H_t + i\omega\epsilon E_t &= g_t \\ \operatorname{curl} H_t + i\omega\epsilon E_3 &= g_3 \,. \end{aligned} \right. \tag{1.1}$$

Eigensystems

Let $E = (\underbrace{E_1, E_2}_{=E_t}, E_3)$, $e_z = (0, 0, 1)$. We will use the 2D identities:

$$\begin{aligned} e_z \times (e_z \times E_t) &= -E_t \\ e_z \times (\nabla \times E_3) &= \nabla E_3 & \qquad e_z \times \nabla E_3 &= -\nabla \times E_3 \\ \operatorname{curl}(e_z \times E_t) &= \operatorname{div} E_t & \qquad \operatorname{div}(e_z \times E_t) &= -\operatorname{curl} E_t \,. \end{aligned}$$

The original system of equations,

$$\nabla \times E - i\omega H = f \qquad \nabla \times H + i\omega\epsilon E = g$$

Multiplying the first and third equations by $i\omega\, e_z \times$, we obtain:

$$\left\{ \begin{aligned} \nabla i\omega E_3 - \frac{\partial}{\partial z} i\omega E_t + \omega^2\, e_z \times H_t &= i\omega\, e_z \times f_t \\ \operatorname{curl} E_t - i\omega H_3 &= f_3 \\ \nabla i\omega H_3 - \frac{\partial}{\partial z} i\omega H_t - \omega^2\, e_z \times \epsilon E_t &= i\omega\, e_z \times g_t \\ \operatorname{curl} H_t + i\omega\epsilon E_3 &= g_3 \,. \end{aligned} \right. \tag{1.1}$$

Waveguide Problem and Its Adjoint

The eigensystem corresponding to the first order system operator, and $e^{i\beta z}$ ansatz in z:

$$\left\{\begin{aligned} E_t \in H_0(\mathrm{curl}, D),\ E_3 \in H_0^1(D) \\ H_t \in H(\mathrm{curl}, D),\ H_3 \in H^1(D) \\ i\omega\nabla E_3 + \omega^2 e_z \times H_t &= -\omega\beta E_t \\ \mathrm{curl}\, E_t - i\omega H_3 &= 0 \\ i\omega\nabla H_3 - \omega^2 e_z \times \epsilon E_t &= -\omega\beta H_t \\ \mathrm{curl}\, H_t + i\omega\epsilon E_3 &= 0\,. \end{aligned}\right. \tag{1.2}$$

The system corresponding to the adjoint:

$$\left\{\begin{aligned} F_t \in H(\mathrm{div}, D),\ F_3 \in H^1(D) \\ G_t \in H_0(\mathrm{div}, D),\ G_3 \in H_0^1(D) \\ \nabla \times F_3 + \omega^2 e_z \times \epsilon G_t &= -\omega\beta F_t \\ i\omega(\mathrm{div}\, F_t - \epsilon G_3) &= 0 \\ \nabla \times G_3 - \omega^2 e_z \times F_t &= -\omega\beta G_t \\ i\omega(\mathrm{div}\, G_t + F_3) &= 0\,. \end{aligned}\right. \tag{1.3}$$

Eigensystems

TEXAS
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Eliminating E_3 and H_3 from system (1.2), we obtain a simplified but second order "EH system" for E_t, H_t only.

$$\left\{\begin{aligned} E_t \in H_0(\mathrm{curl}, D),\ \mathrm{curl}\, E_t \in H^1(D) \\ H_t \in H(\mathrm{curl}, D),\ \tfrac{1}{\epsilon}\,\mathrm{curl}\, H_t \in H_0^1(D) \\ -\boldsymbol{\nabla}(\tfrac{1}{\epsilon}\,\mathrm{curl}\, H_t) + \omega^2 \boldsymbol{e_z} \times H_t &= -\omega\beta E_t \\ \boldsymbol{\nabla}(\mathrm{curl}\, E_t) - \omega^2 \boldsymbol{e_z} \times \epsilon E_t &= -\omega\beta H_t\,. \end{aligned}\right. \tag{1.4}$$

Similarly, eliminating F_3 and G_3 from system (1.3), we obtain a second order "FG system" for F_t, G_t only.

$$\left\{\begin{aligned} F_t \in H(\mathrm{div}, D),\ \tfrac{1}{\epsilon}\,\mathrm{div}\, F_t \in H_0^1(D) \\ G_t \in H_0(\mathrm{div}, D),\ \mathrm{div}\, G_t \in H^1(D) \\ -\boldsymbol{\nabla} \times \mathrm{div}\, G_t + \omega^2 \boldsymbol{e_z} \times \epsilon G_t &= -\omega\beta F_t \\ \boldsymbol{\nabla} \times (\tfrac{1}{\epsilon}\,\mathrm{div}\, F_t) - \omega^2 \boldsymbol{e_z} \times F_t &= -\omega\beta G_t\,. \end{aligned}\right. \tag{1.5}$$

One can check that the operator in (1.5) corresponds to the adjoint of operator in (1.4). Notice how the BCs on E_3, G_3 have been inherited by $\mathrm{curl}\, H_t$ and $\mathrm{div}\, F_t$.

Reduction to single variable eigensystems

Assume $\beta \neq 0$. Solving $(1.4)_2$ for H_t,

$$\begin{aligned} H_t &= -\frac{1}{\omega\beta}[\nabla \operatorname{curl} E_t - \omega^2 e_z \times \epsilon E_t] \\ \operatorname{curl} H_t &= \frac{\omega}{\beta} \operatorname{curl}(e_z \times \epsilon E_t) = \frac{\omega}{\beta} \operatorname{div} \epsilon E_t \end{aligned} \tag{1.6}$$

and substituting it into $(1.4)_1$, we obtain an ``E eigenvalue problem'' for E_t alone.

$$\begin{cases} E_t \in H_0(\operatorname{curl}, D),\ \operatorname{curl} E_t \in H^1(D),\ \frac{1}{\epsilon} \operatorname{div} \epsilon E_t \in H_0^1(D) \\ \nabla \times \operatorname{curl} E_t - \omega^2 \epsilon E_t - \nabla(\frac{1}{\epsilon} \operatorname{div} \epsilon E_t) = -\beta^2 E_t\,. \end{cases} \tag{1.7}$$

Similarly, solving $(1.4)_1$ for E_t,

$$\begin{aligned} E_t &= -\frac{1}{\omega\beta}[-\nabla(\frac{1}{\epsilon} \operatorname{curl} H_t) + \omega^2 e_z \times H_t] \\ \operatorname{curl} E_t &= -\frac{\omega}{\beta} \operatorname{curl}(e_z \times H_t) = -\frac{\omega}{\beta} \operatorname{div} H_t \end{aligned} \tag{1.8}$$

and substituting it into $(1.4)_2$, we obtain an ``H eigenvalue problem'' for H_t alone.

$$\begin{cases} H_t \in H(\operatorname{curl}, D) \cap H_0(\operatorname{div}, D),\ \frac{1}{\epsilon} \operatorname{curl} H_t \in H_0^1(D),\ \operatorname{div} H_t \in H^1(D) \\ \epsilon \nabla \times (\frac{1}{\epsilon} \operatorname{curl} H_t) - \omega^2 \epsilon H_t - \nabla(\operatorname{div} H_t) = -\beta^2 H_t\,. \end{cases} \tag{1.9}$$

Note that BC: $n \times E_t = 0$ implies BC: $n \cdot H_t = 0$.

Same for the Adjoint Problem

Solving $(1.5)_2$ for G_t,

$$\begin{aligned} G_t &= -\frac{1}{\omega\beta}[\nabla \times (\frac{1}{\epsilon}\operatorname{div} F_t) - \omega^2 e_z \times F_t] \\ \operatorname{div} G_t &= \frac{\omega}{\beta}\operatorname{div}(e_z \times F_t) = -\frac{\omega}{\beta}\operatorname{curl}\epsilon F_t \end{aligned} \qquad (1.10)$$

and substituting it into $(1.5)_1$, we obtain an "F eigenvalue problem" for F_t alone.

$$\left\{ \begin{array}{l} F_t \in H_0(\operatorname{curl}, D) \cap H(\operatorname{div}, D),\ \frac{1}{\epsilon}\operatorname{div} F_t \in H_0^1(D),\ \operatorname{curl} F_t \in H^1(D) \\ \nabla \times \operatorname{curl} F_t - \omega^2 \epsilon F_t - \epsilon\nabla(\frac{1}{\epsilon}\operatorname{div} F_t) = -\gamma^2 F_t\,. \end{array} \right. \qquad (1.11)$$

Note that BC: $n \cdot G_t = 0$ implies BC: $n \times F_t = 0$. Similarly, solving $(1.5)_1$ for F_t,

$$\begin{aligned} F_t &= -\frac{1}{\omega\beta}[-\nabla \times \operatorname{div} G_t) + \omega^2 e_z \times \epsilon G_t] \\ \operatorname{div} F_t &= -\frac{\omega}{\beta}\operatorname{div}(e_z \times \epsilon G_t) = -\frac{\omega}{\beta}\operatorname{curl}\epsilon G_t \end{aligned} \qquad (1.12)$$

and substituting it into $(1.5)_2$, we obtain an "G eigenvalue problem" for G_t alone.

$$\left\{ \begin{array}{l} G_t \in H_0(\operatorname{div}, D),\ \operatorname{div} G_t \in H^1(D),\ \frac{1}{\epsilon}\operatorname{curl}\epsilon G_t \in H_0^1(D) \\ \nabla \times (\frac{1}{\epsilon}\operatorname{curl}\epsilon G_t) - \omega^2 \epsilon G_t - \nabla(\operatorname{div} G_t) = -\gamma^2 G_t\,. \end{array} \right. \qquad (1.13)$$

Lemma (1)

(a) *Let* $((E_t, H_t), -\omega\beta)$ *be an eigenpair for* EH *system (1.4). Then* $(E_t, -\beta^2)$ *solves the* E *problem (1.7), and* $(H_t, -\beta^2)$ *solves the* H *problem (1.9).*

(b) *Conversely, if* $(E_t, -\beta^2)$ *is an eigenpair for the* E *problem (1.7), and we define* H_t *by*

$$H_t = \frac{1}{\omega(\pm\beta)} \left(-\boldsymbol{\nabla} \operatorname{curl} E_t + \omega^2 \boldsymbol{e_z} \times \epsilon E_t\right)$$

then $((E_t, H_t), -\omega(\pm\beta))$ *is an eigenpair for* EH *system (1.4). Each eigenpair for* E *problem (1.7) generates two eigenpairs for* EH *problem (1.4).*

(c) *Similarly, if* $(H_t, -\beta^2)$ *is an eigenpair for* H *problem (1.9), and we define* E_t *by:*

$$E_t = \frac{1}{\omega(\pm\beta)} \left(\boldsymbol{\nabla}\left(\frac{1}{\epsilon} \operatorname{curl} H_t\right) - \omega^2 \boldsymbol{e_z} \times H_t\right)$$

then $((E_t, H_t), -\omega(\pm\beta))$ *is an eigenpair for* EH *system (1.4). Each eigenpair for* H *problem (1.9) generates two eigenpairs for* EH *problem (1.4).*

In particular, Lemma 1 implies that E and H eigenproblems have the same eigenvalues β^2.

Lemma (2)

(a) *Let $((F_t, G_t), \omega\gamma)$ be an eigenpair for FG system (1.5). Then $(G_t, -\gamma^2)$ solves G problem (1.13) and $(F_t, -\gamma^2)$ solves F problem (1.11).*

(b) *Conversely, if $(F_t, -\gamma^2)$ is an eigenpair for F problem (1.11), and we define G_t by*

$$G_t = \frac{1}{\omega(\pm\gamma)} \left(-\nabla \operatorname{curl} F_t + \omega^2 \boldsymbol{e}_z \times \epsilon F_t\right)$$

then $((F_t, G_t), \omega(\pm\gamma))$ is an eigenpair for FG system (1.5). Each eigenpair for F problem (1.11) generates two eigenpairs for FG system (1.5).

(c) *Similarly, if $(G_t, -\gamma^2)$ is an eigenpair for G problem (1.13), and we define F_t by:*

$$F_t = \frac{1}{\omega(\pm\gamma)} \left(\nabla\left(\frac{1}{\epsilon} \operatorname{curl} G_t\right) - \omega^2 \boldsymbol{e}_z \times G_t\right)$$

then $((F_t, G_t), \omega(\pm\gamma))$ is an eigenpair for FG system (1.5). Each eigenpair for G problem (1.13) generates two eigenpairs for FG system (1.5).

In particular, Lemma 2 implies that F and G eigenproblems have the same eigenvalues γ^2.

Relation Between Eigensystems and the Adjoint Eigensystems

Lemma (3)

$(E_t, -\beta^2)$ is an eigenpair for E problem (1.7) if an only if $(G_t := e_z \times E_t, -\beta^2)$ is an eigenpair for G problem (1.13). Similarly, $(H_t, -\beta^2)$ is an eigenpair for H problem (1.9) if and only if $(F_t := e_z \times H_t, -\beta^2)$ is an eigenpair for F problem (1.11). In particular, this implies that all four individual eigenproblems share the same eigenvalues.

Outline

Homogeneous Waveguide

For $\epsilon = 1$, the E problem (1.7) reduces to ($E = E_t$):

$$\begin{cases} E \in H_0(\mathrm{curl}, D), \ \mathrm{curl}\, E \in H^1(D), \ \mathrm{div}\, E \in H_0^1(D) \\ \underbrace{\nabla \times \mathrm{curl}\, E - \omega^2 E - \nabla(\mathrm{div}\, E)}_{=:AE} = -\beta^2 E\,. \end{cases} \tag{2.14}$$

We get the same equation for the H problem (1.9) but with different BCs ($H = H_t$):

$$H \in H(\mathrm{curl}, D) \cap H_0(\mathrm{div}, D), \ \mathrm{curl}\, H \in H_0^1(D), \ \mathrm{div}\, H \in H^1(D)\,.$$

Lemma (4. Helmholtz decompositions)

Let $D \subset \mathbb{R}^2$ be a simply connected domain. For every $E \in L^2(D)^2$ there exist a unique $\phi \in H_0^1(D)$ and a unique $\psi \in H^1(D)$, $\int_D \psi = 0$, such that

$$E = \nabla\phi + \nabla \times \psi\,. \tag{2.15}$$

Similarly, for every $H \in L^2(D)^2$ there exist a unique $\phi \in H_0^1(D)$ and a unique $\psi \in H^1(D)$, $\int_D \psi = 0$, such that

$$H = \nabla \times \phi + \nabla\psi\,. \tag{2.16}$$

Homogeneous Waveguide

Consider now the eigenvalue problem (2.14) and Helmholtz decomposition of E. Boundary condition $E_t = 0$ implies that $\frac{\partial \psi}{\partial n} = 0$ on ∂D. Substituting (2.15) into (2.14), we obtain:

$$\nabla \times \underbrace{\left(-\Delta\psi + (\beta^2 - \omega^2)\psi\right)}_{=:\Psi} + \nabla \underbrace{\left(-\Delta\phi + (\beta^2 - \omega^2)\phi\right)}_{=:\Phi} = 0\,. \tag{2.17}$$

The equation above represents the Helmholtz decomposition of zero function. Uniqueness of ϕ and ψ in the Helmholtz decomposition implies now that $\Phi = \Psi = 0$. Let (λ_i, ϕ_i) and (μ_j, ψ_j) be the Dirichlet and Neumann eigenpairs of the Laplacian in domain D. Vanishing of Φ and Ψ implies that there exist i, j such that

$$\phi = \phi_i,\ \omega^2 - \beta^2 = \lambda_i \quad \text{and} \quad \psi = \psi_j, \omega^2 - \beta^2 = \mu_j\,.$$

If the Dirichlet and Neumann eigenvalues are distinct, eigenvector E must reduce to either gradient or curl. This is the case, e.g., for a circular domain D. In the case of a common Dirichlet and Neumann eigenvalue, $\lambda_i = \mu_j$, we obtain a multiple eigenvalue $\beta^2 = \omega^2 - \lambda_i = \omega^2 - \mu_j$, with the eigenspace consisting of vectors:

$$E = A\nabla \times \psi_j + B\nabla\phi_i\,, \quad A, B \in \mathbb{C}\,.$$

Lemma (5)

Let (λ_i, ϕ_i) and (μ_j, ψ_j) denote the Dirichlet and Neumann eigenpairs of the Laplacian in domain D. The eigenvalues β_i^2 are classified into the following three families.

(a) $\beta^2 = \omega^2 - \mu_j$ *with* μ_j *distinct from all* λ_i*. The corresponding eigenvectors are curls:*

$$\boldsymbol{E} = \boldsymbol{\nabla} \times \psi_j \,,$$

with multiplicity of β^2 *equal to the multiplicity of* μ_j*.*

(a) $\beta^2 = \omega^2 - \lambda_i$ *with* λ_i *distinct from all* μ_j*. The corresponding eigenvectors are gradients:*

$$\boldsymbol{E} = \boldsymbol{\nabla} \phi_i \,,$$

with multiplicity of β^2 *equal to the multiplicity of* λ_i*.*

(c) $\beta^2 = \omega^2 - \mu_j = \omega^2 - \lambda_i$ *for* $\mu_j = \lambda_i$*. The corresponding eigenvectors are linear combinations of curls and gradients:*

$$\boldsymbol{E} = A\boldsymbol{\nabla} \times \psi_j + B\boldsymbol{\nabla}\phi_i \,, \quad A, B \in \mathbb{C} \,,$$

with multiplicity of β^2 *equal to the sum of multiplicities of* μ_j *and* λ_i*.*

Lemma (6)

Let (λ_i, ϕ_i) and (μ_j, ψ_j) denote the Dirichlet and Neumann eigenpairs of the Laplacian in domain D. The eigenvalues γ_i^2 are classified into the following three families.

(a) $\gamma^2 = \omega^2 - \mu_j$ *with* μ_j *distinct from all* λ_i*. The corresponding eigenvectors are gradients:*

$$H = \boldsymbol{\nabla} \psi_j \,,$$

with multiplicity of β^2 *equal to the multiplicity of* μ_j*.*

(a) $\gamma^2 = \omega^2 - \lambda_i$ *with* λ_i *distinct from all* μ_j*. The corresponding eigenvectors are curls:*

$$H = \boldsymbol{\nabla} \times \phi_i \,,$$

with multiplicity of γ^2 *equal to the multiplicity of* λ_i*.*

(c) $\gamma^2 = \omega^2 - \mu_j = \omega^2 - \lambda_i$ *for* $\mu_j = \lambda_i$*. The corresponding eigenvectors are linear combinations of gradients and curls:*

$$H = A\boldsymbol{\nabla} \psi_j + B\boldsymbol{\nabla} \times \phi_i \,, \quad A, B \in \mathbb{C} \,,$$

with multiplicity of γ^2 *equal to the sum of multiplicities of* μ_j *and* λ_i*.*

Outline

Cylindrical Waveguide

Consider the Dirichlet or Neumann Laplace eigenvalue problem in a unit circle,

$$-\Delta u = \lambda u, \quad \lambda = \nu^2 .$$

For Dirichlet problem the operator is positive definite, so $\nu > 0$, for Neumann problem, $u = \text{const}$ corresponds to zero eigenvalue, all other eigenvalues are positive as well. Rewriting the operator in polar coordinates r, θ,

$$-\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \nu^2 u .$$

Separating the variables, $u = R(r)\Theta(\theta)$, we get

$$-\frac{1}{r}(rR')'\Theta - \frac{1}{r^2}R\Theta'' = \nu^2 R\Theta$$

or,

$$\frac{r(rR')'}{R} + \nu^2 r^2 = -\frac{\Theta''}{\Theta} = k^2$$

where k^2 is a real and positive separation constant. We obtain,

$$\Theta = A\cos k\theta + B\sin k\theta$$

and the periodic BCs on u and, therefore, Θ, imply that $k = 0, 1, 2, \ldots$.

Cylindrical Waveguide

This leads to the Bessel equation in r,

$$r(rR')' + (\nu^2 r^2 - k^2)R = 0$$

with solution:

$$R = CJ_k(\nu r) + DY_k(\nu r)\,.$$

Finite energy condition eliminates the second term, $D = 0$.

Dirichlet BC: $R(1) = 0$ leads to ν being a root of the Bessel function $J_k(\nu) = 0$. We have a family of roots (and, therefore Dirichlet Laplace eigenvalues ν^2) : $\nu = \nu_{k,m},\ k = 0, 1, 2, \ldots, m = 1, 2, \ldots$. For $k = 0$, the roots are simple, with corresponding eigenvectors given by:

$$u = J_0(\nu_{0,m} r)\,.$$

For $k > 0$, we have double eigenvectors with eigenspaces given by:

$$u = J_0(\nu_{k,m} r)(A\cos k\theta + B\sin k\theta)\,.$$

Neumann BC: The situation is similar except that we are dealing now with the roots of the derivative of Bessel functions: $J'_k(\lambda) = 0$, $\lambda = \lambda_{k,m},\ k = 0, 1, 2, \ldots, m = 1, 2, \ldots$.

Cylindrical Waveguide - Cont.

m/k	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

Roots of Bessel functions $\nu_{k,m}$.

m/k	$J_0'(x)$	$J_1'(x)$	$J_2'(x)$	$J_3'(x)$	$J_4'(x)$	$J_5'(x)$
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

Roots of derivatives of Bessel functions $\lambda_{k,m}$.

Outline

Perturbation Analysis

The E problem (1.7):

$$\nabla \times \operatorname{curl} E_t - \omega^2 \epsilon E_t - \nabla(\frac{1}{\epsilon} \operatorname{div} \epsilon E_t) \quad = -\beta^2 E_t$$

is not self-adjoint, but it is a perturbation of the self-adjoint homogeneous E problem for $\epsilon = 1$. The homogeneous problem admits two families of eigenvectors:

$$\begin{aligned} E_i &= \nabla \times \psi_i, & \beta_i^2 &= \omega^2 - \mu_i \\ E_j &= \nabla \phi_j, & \beta_j^2 &= \omega^2 - \lambda_j \end{aligned}$$

where (μ_i, ψ_i) and (λ_j, ϕ_j) are Neumann and Dirichlet eigenpairs for the Laplace operator. Consider now a perturbation,

$$\epsilon = 1 + \delta\epsilon, \qquad E := E + \delta E, \quad \beta^2 := \beta^2 + \delta\beta^2 \,.$$

Plugging the perturbations into the E problem and linearizing, we obtain the corresponding linearized problem:

$$A(\delta E_t) + \beta^2 \delta E_t = \omega^2 \delta\epsilon E - \nabla(\delta\epsilon \operatorname{div} E) + \nabla \operatorname{div}(\delta\epsilon E) - \delta\beta^2 E \,.$$

Perturbation Analysis - Cont.

Consider now the homogeneous and perturbed E problems for a specific eigenpair $(-\beta_i^2, E_i)$. Representing the perturbation in eigenbasis E_j, we have:

$$\begin{aligned} \delta E_i &= \sum_j (\delta E_i, E_j) E_j \\ A(\delta E_i) &= \sum_j (\delta E_i, E_j)(-\beta_j^2) E_j \\ (A(\delta E_i), E_k) &= \sum_j (-\beta_j^2)(\delta E_i, E_j) \underbrace{(E_j, E_k)}_{=\delta_{jk}} = (-\beta_k^2)(\delta E_i, E_k)\,. \end{aligned}$$

Taking the L^2-product of the linearized perturbed problem with E_k, we obtain:

$$(\beta_i^2 - \beta_k^2)(\delta E_i, E_k) + \delta\beta_i^2 \delta_{ik} = \omega^2(\delta\epsilon E_i, E_k) - (\boldsymbol{\nabla}(\delta\epsilon \operatorname{div} E_i), E_k) + (\boldsymbol{\nabla} \operatorname{div}(\delta\epsilon E_i), E_k)\,.$$

Under the assumption of distinct (simple) eigenvalues, for $k = i$, we get a formula for perturbation $\delta\beta_i^2$,

$$\delta\beta_i^2 = \omega^2(\delta\epsilon E_i, E_i) + (\delta\epsilon \operatorname{div} E_i, \operatorname{div} E_i) - (\operatorname{div}(\delta\epsilon E_i), \operatorname{div} E_i)\,.$$

For $k \neq i$, the formula allows to compute perturbation δE_i; the i-th component of δE_i comes from the normalization $\|E_i + \delta E_i\| = 1$.

$$(\beta_i^2 - \beta_k^2)(\delta E_i, E_k) = \omega^2(\delta\epsilon E_i, E_k) + (\delta\epsilon \operatorname{div} E_i, \operatorname{div} E_k) - (\operatorname{div}(\delta\epsilon E_i), \operatorname{div} E_k)\,.$$

Linearized Mass Matrices

Mass term $(\delta E, E)$ for different families of eigenvectors:

$(\delta E, E)$	$E_k = \nabla \times \psi_k$	$E_l = \nabla \phi_l$
$\delta E_i = \delta(\nabla \times \psi_i)$	$\dfrac{\omega^2(\delta\epsilon E_i, E_k)}{\mu_k - \mu_i}$	$\dfrac{(\omega^2 - \lambda_l)(\delta\epsilon E_i, E_l)}{\lambda_l - \mu_i}$
$\delta E_j = \delta(\nabla \phi_j)$	$\dfrac{\omega^2(\delta\epsilon E_j, E_k)}{\mu_k - \lambda_j}$	$\dfrac{(\omega^2 - \lambda_l)(\delta\epsilon E_j, E_l) + \lambda_j\lambda_l(\delta\epsilon\phi_j, \phi_l)}{\lambda_l - \lambda_j}$

Linearized mass matrix $(\delta E_i, E_k) + (E_i, \delta E_k)$ for different families of eigenvectors.

$(\delta E, E) + (E, \delta E)$	$\delta E_k = \delta(\nabla \times \psi_k)$	$\delta E_l = \delta(\nabla \phi_l)$
$\delta E_i = \delta(\nabla \times \psi_i)$	0	not needed
$\delta E_j = \delta(\nabla \phi_j)$	not needed	$-(\delta\epsilon E_j, E_l)$

Linearized Curl-Curl Mass Matrix

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We have:

$$
\begin{aligned}
\delta E_i &= \sum_k (\delta E_i, E_k) E_k && \text{(summation over both curls and grads)} \\
\operatorname{curl} \delta E_i &= \sum_k (\delta E_i, \nabla \times \psi_k) \mu_k \psi_k && \text{(summation over curls only.)}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(\operatorname{curl} \delta E_i, \operatorname{curl} E_j) &= (\textstyle\sum_k (\delta E_i, \nabla \times \psi_k) \mu_k \psi_k, \operatorname{curl} E_j) \\
&= \textstyle\sum_k (\delta E_i, \nabla \times \psi_k)(\mu_k \psi_k, \mu_j \psi_j) \\
&= (\delta E_i, \nabla \times \psi_j) \mu_j
\end{aligned}
$$

is non-zero only if E_j is a curl, $E_j = \nabla \times \psi_j$.
Consequently, the linearized curl-curl mass matrix is equal to:

$$
(\delta E_i, E_j)\mu_j + (E_i, \delta E_j)\mu_i = \mu_j \frac{\omega^2 (\delta\epsilon E_i, E_j)}{\mu_j - \mu_i} + \mu_i \frac{\omega^2 (\delta\epsilon E_i, E_j)}{\mu_i - \mu_j} = \omega^2 (\delta\epsilon E_i, E_j)
$$

if $E_i = \nabla \times \psi_i, E_j = \nabla \times \psi_j$.

Outline

Reduction to the Second Order System

We return to the original first order system. Testing the first equation with F_t, and the third equation with G_t, $n \cdot G_t = 0$ on ∂D, to obtain:

$$\left\{ \begin{aligned} -(i\omega E_3, \operatorname{div} F_t) + \omega^2 (e_z \times H_t, F_t) - \tfrac{\partial}{\partial z} i\omega (E_t, F_t) &= i\omega\,(e_z \times f_t, F_t) \\ \operatorname{curl} E_t - i\omega H_3 &= f_3 \\ -(i\omega H_3, \operatorname{div} G_t) - \omega^2 (e_z \times \epsilon E_t, G_t) - \tfrac{\partial}{\partial z} i\omega (H_t, G_t) &= i\omega\,(e_z \times g_t, G_t) \\ \operatorname{curl} H_t + i\omega\epsilon E_3 &= g_3\,. \end{aligned} \right.$$

Note that, when integrating by parts the first terms, we have used the fact that $E_3 = 0$ and $n \cdot G_t = 0$ on ∂D. Solving the second and fourth equations in (1.1) for E_3 and H_3,

$$E_3 = \frac{1}{i\omega\epsilon} g_3 - \frac{1}{i\omega\epsilon} \operatorname{curl} H_t \qquad H_3 = -\frac{1}{i\omega} f_3 + \frac{1}{i\omega} \operatorname{curl} E_t\,,$$

and substituting into the first and the third equations, we obtain a system of two variational equations for E_t, H_t:

$$\left\{ \begin{aligned} (\tfrac{1}{\epsilon} \operatorname{curl} H_t, \operatorname{div} F_t) + \omega^2 (e_z \times H_t, F_t) - \tfrac{\partial}{\partial z} i\omega (E_t, F_t) &= i\omega\,(e_z \times f_t, F_t) + (\tfrac{1}{\epsilon} g_3, \operatorname{div} F_t) \\ -(\operatorname{curl} E_t, \operatorname{div} G_t) - \omega^2 (e_z \times \epsilon E_t, G_t) - \tfrac{\partial}{\partial z} i\omega (H_t, G_t) &= i\omega\,(e_z \times g_t, G_t) - (f_3, \operatorname{div} G_t)\,. \end{aligned} \right. \tag{5.18}$$

Decoupling

Variational eigenvalue problem:

$$\begin{cases} E_t \in H_0(\mathrm{curl}, D), H_t \in H(\mathrm{curl}, D) \\ (\frac{1}{\epsilon}\,\mathrm{curl}\, H_t, \mathrm{div}\, F_t) + \omega^2(e_z \times H_t, F_t) = -\omega\beta(E_t, F_t) \\ -(\mathrm{curl}\, E_t, \mathrm{div}\, G_t) - \omega^2(e_z \times \epsilon E_t, G_t) = -\omega\beta(H_t, G_t) \\ \qquad F_t \in H(\mathrm{div}, D),\ G_t \in H_0(\mathrm{div}, D)\,, \end{cases}$$

is equivalent to the EH eigenproblem. Similarly, switching the role of (E_t, H_t) and (F_t, G_t) above, we obtain the adjoint variational eigenvalue problem equivalent to the FG eigenproblem.
We expand the unknowns into series of the perturbed eigenvectors:

$$\begin{aligned} E_t &= \textstyle\sum_i \alpha_i E_{t1,i} + \sum_j \beta_j E_{t2,j} \\ H_t &= \textstyle\sum_i \delta_i H_{t1,i} + \sum_j \eta_j H_{t2,j} \end{aligned}$$

where $\alpha_i, \beta_j, \delta_i, \eta_j$ are functions of z, and

$$\begin{aligned} E_{t1,i} &= \nabla \times \psi_i + \delta E_{t1,i}, & E_{t2,j} &= \nabla \phi_j + \delta E_{t2,j} \\ H_{t1,i} &= \nabla \psi_i + \delta H_{t1,i}, & H_{t2,j} &= \nabla \times \phi_j + \delta H_{t2,j} \end{aligned}$$

are the two E and H families of (perturbed) eigenvectors.

Decoupling

Let

$$F_{t1,i} = \nabla \times \psi_i + \delta F_{t1,i}, \qquad F_{t2,j} = \nabla \phi_j + \delta F_{t2,j}$$

$$G_{t1,i} = \nabla \psi_i + \delta G_{t1,i}, \qquad G_{t2,j} = \nabla \times \phi_j + \delta G_{t2,j}$$

be the corresponding families of perturbed adjoint eigenvectors.

Scalings:

$$\|\nabla \times \psi_i\| = \|\nabla \psi_i\| = 1, \quad (\delta E_{t1,i}, \nabla \times \psi_i) = 0, \quad (\delta F_{t1,i}, \nabla \times \psi_i) = 0 \quad \Rightarrow$$

$$\|\nabla \times \psi_i + \delta E_{t1,i}\| = 1 \quad \text{and} \quad (\nabla \times \psi_i + \delta E_{t1,i}, \nabla \times \psi_i + \delta F_{t1,i}) = 1\,.$$

Same for the H and G eigenvectors, and the second family of eigenvectors.

Let $-\beta^2$ be an eigenvalue for E and H eigenproblems with the corresponding eigenvectors E_t, H_t scaled as above. In order to invoke Lemma 1 (b) argument, we have to replace H_t with cH_t where constant c is computed by comparing eigenvector cH_t with H_t given by relation (1.6),

$$cH_t = \frac{1}{\omega\beta}[-\nabla \operatorname{curl} E_t + \omega^2 e_z \times \epsilon E_t]\,.$$

Pair (E_t, cH_t) constitutes then an eigenvector for system (1.4) corresponding to root β of β^2 selected in such a way that $e^{i\beta z}$ represents an outgoing wave. We proceed similarly with the adjoint eigenvectors. Let $-\gamma^2$ be an eigenvalue for problems (1.11) and (1.13) with the corresponding eigenvectors F_t, H_t. After scaling the second component, pair (F_t, dG_t) constitutes an eigenvector for system (1.5) corresponding to a root γ of γ^2.

Decoupling

Case: $\beta^2 \neq \gamma^2$ and, therefore, $\beta \neq \gamma$. Testing the 2nd order system with pair (F_t, G_t), we obtain the bi-orthogonality condition,

$$c(BH_t, F_t) + d(CE_t, G_t) = 0$$

where B and C denote the operators on the left-hand side of the system. But, testing with the adjoint eigenpair $(F_t, -G_t)$ (corresponding to eigenvalue $-\gamma \neq \beta$), we obtain also

$$c(BH_t, F_t) - d(CE_t, G_t) = 0\,.$$

Consequently,

$$(BH_t, F_t) = 0 \qquad \text{and} \qquad (CE_t, G_t) = 0\,.$$

Case: $\beta^2 = \gamma^2$ and $\beta = \gamma$. Testing with pair (F_t, dH_t), we obtain:

$$c(BH_t, F_t) + d(CE_t, G_t) = \omega\beta[1 + cd]\,.$$

But, testing with the adjoint eigenpair $(F_t, -H_t)$ (corresponding to eigenvalue $-\gamma \neq \beta$), we obtain also

$$c(BH_t, F_t) - d(CE_t, G_t) = 0$$

Consequently,

$$(BH_t, F_t) = -\frac{\omega\beta}{2c}[1 + cd] =: \theta \qquad \text{and} \qquad (CE_t, G_t) = -\frac{\omega\beta}{2d}[1 + cd] =: \nu\,.$$

Decoupling

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Theorem

Testing with $(F_{t1,j}, G_{t1,j})$ and with $(F_{t2,j}, G_{t2,j})$ we obtain a decoupled system of ODEs for the coefficients α_j, δ_j:

$$\left\{ \begin{aligned} \theta_{1,j}\delta_j - i\omega\alpha_j' &= r_1(z) := (i\omega\, e_z \times f_t, F_{t1,j}) + (\frac{1}{\epsilon} g_3, \operatorname{div} F_{t1,j}) \\ \nu_{1,j}\alpha_j - i\omega\delta_j' &= r_2(z) := (i\omega\, e_z \times g_t, G_{t1,j}) - (f_3, \operatorname{div} G_{t1,j}) \end{aligned} \right. \tag{5.19}$$

and β_j, η_j:

$$\left\{ \begin{aligned} \theta_{2,j}\eta_j - i\omega\beta_j' &= s_1(z) := (i\omega\, e_z \times f_t, F_{t2,j}) + (\frac{1}{\epsilon} g_3, \operatorname{div} F_{t2,j}) \\ \nu_{2,j}\beta_j - i\omega\eta_j' &= s_2(z) := (i\omega\, e_z \times g_t G_{t2,j}) - (f_3, \operatorname{div} G_{t2,j}) \end{aligned} \right. \tag{5.20}$$

where

$$\begin{aligned} \theta_{1,j} &= -\omega^2, & \nu_{1,j} &= -\beta_j^2 - \omega^2(\delta\epsilon \nabla\psi_j, \nabla\psi_j) \\ \theta_{2,j} &= \beta_j^2 + \lambda_j^2(\delta\epsilon\, \phi_j, \phi_j) & \nu_{2,j} &= \omega^2 + \omega^2(\delta\epsilon\, \nabla\phi_j, \nabla\phi_j)\,. \end{aligned}$$

Watch out for the terrible notational collision with β's.

Outline

Estimation of E_t

$$
\begin{aligned}
\|E_t\|^2 &\leq 2\left[\|\textstyle\sum_{i=1}^{\infty} \alpha_i E_{t1,i}\|^2 + \|\sum_{j=1}^{\infty} \beta_j E_{t2,j}\|^2\right] \\
&= 2\lim_{N\to\infty}\left[(\textstyle\sum_{i=1}^{N} \alpha_i E_{t1,i}, \sum_{k=1}^{N} \alpha_k E_{t1,k}) + (\sum_{j=1}^{N} \beta_j E_{t2,j}, \sum_{l=1}^{N} \beta_l E_{t2,l})\right] \\
&= 2\lim_{N\to\infty}\left[\textstyle\sum_{i,k=1}^{N} \alpha_i \overline{\alpha_k}(E_{t1,i}, E_{t1,k}) + \sum_{j,l=1}^{N} \beta_j \overline{\beta_l}(E_{t2,j}, E_{t2,l})\right] \\
&\leq \lim_{N\to\infty} C\left[\textstyle\sum_{i=1}^{N} |\alpha_i|^2 + \sum_{j=1}^{N} |\beta_j|^2\right] \\
&= C\left[\textstyle\sum_{i=1}^{\infty} |\alpha_i|^2 + \sum_{j=1}^{\infty} |\beta_j|^2\right]
\end{aligned}
$$

where (up to linearization) $C = 2(1 + \|\delta\epsilon\|_{L^\infty(D)})$. After integrating in z, we get

$$
\int_0^L \|E_t\|^2\, dz \leq C\left[\sum_{i=1}^{\infty} \int_0^L |\alpha_i|^2\, dz + \sum_{j=1}^{\infty} \int_0^L |\beta_j|^2\, dz\right].
$$

Estimation of $\operatorname{curl} E_t$

$$
\begin{aligned}
\|\operatorname{curl} E_t\|^2 &\leq 2\left[\|\textstyle\sum_{i=1}^{\infty}\alpha_i \operatorname{curl} E_{t1,i}\|^2 + \|\sum_{j=1}^{\infty}\beta_j \operatorname{curl} E_{t2,j}\|^2\right] \\
&= 2\lim_{N\to\infty}\Big[(\textstyle\sum_{i=1}^{N}\alpha_i \operatorname{curl} E_{t1,i}, \sum_{k=1}^{N}\alpha_k \operatorname{curl} E_{t1,k}) \\
&\qquad\qquad + (\textstyle\sum_{j=1}^{N}\beta_j \operatorname{curl} E_{t2,j}, \sum_{l=1}^{N}\beta_l \operatorname{curl} E_{t2,l})\Big] \\
&= 2\lim_{N\to\infty}\left[\textstyle\sum_{i,k=1}^{N}\alpha_i\overline{\alpha_k}(\operatorname{curl} E_{t1,i}, \operatorname{curl} E_{t1,k}) + \sum_{j,l=1}^{N}\beta_j\overline{\beta_l}(\operatorname{curl} E_{t2,j}, \operatorname{curl} E_{t2,l})\right] \\
&\approx 2\textstyle\sum_{i=1}^{\infty}(\mu_i + \omega^2\|\delta\epsilon\|_{L^\infty(D)})|\alpha_i|^2 \,.
\end{aligned}
$$

Note that, like for the homogeneous case, the perturbed gradients do not contribute (the linearized perturbed curl mass matrix is zero).

Estimation of Coefficients α_i, δ_i

We focus now on the ODE boundary-value problem for coefficients α and δ,

$$\begin{cases} \alpha(0) = 0,\ \alpha(L) = \frac{\nu}{\theta}\delta(L) \\ \theta\delta - i\omega\alpha' = r_1 \\ \nu\alpha - i\omega\delta' = r_2 \,. \end{cases}$$

Testing the second equation with $\delta\alpha, \delta\alpha(0) = 0$, integrating the derivative term by parts, and utilizing BC, we obtain:

$$i\omega(\delta, \delta\alpha') = -\omega\beta(\alpha, \delta\alpha) + i\omega\alpha(L)\delta\alpha(L) + (r_2, \delta\alpha)\,.$$

Testing now the first equation with $\delta\alpha'$ and using the formula above, we obtain the ultimate variational problem for coefficient α,

$$\begin{cases} \alpha(0) = 0 \\ (\alpha', \delta\alpha') - \kappa^2(\alpha, \delta\alpha) + \kappa\alpha(L)\delta\alpha(L) = \frac{1}{\omega}(r_1, \delta\alpha') - \frac{\beta}{\omega}(r_2, \delta\alpha) \\ \qquad\qquad \forall\, \delta\alpha \,:\, \delta\alpha(0) = 0 \end{cases}$$

where $\kappa = i\frac{\sqrt{\theta\nu}}{\omega}$.

1D Stability Result

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Lemma (7)

Let $I = (0, L)$. *Consider two problems: Find* $q_1, q_2 \in H^1_{(0}(I) := \{w \in H^1(I) \,:\, w(0) = 0\}$ *such that*

$$\begin{aligned} (q_1', w') + \beta^2 (q_1, w) + \beta q_1(L)\overline{w(L)} &= (f, w) \quad w \in H^1_{(0}(I)\,, \\ (q_2', w') + \beta^2 (q_2, w) + \beta q_2(L)\overline{w(L)} &= (f, w') \quad w \in H^1_{(0}(I)\,. \end{aligned}$$

(i) *If* $\beta \in i\mathbb{R}$ *then,*

$$\begin{aligned} \|q_1'\|^2 + \beta^2 \|q_1\|^2 &\leq C L^2 \|f\|^2\,, \\ \|q_2'\|^2 + \beta^2 \|q_2\|^2 &\leq C L^2 |\beta|^2 \|f\|^2\,, \end{aligned}$$

where $C > 0$ *depend only on a lower bound for* $L|\beta|$.

(ii) *If* $\beta > 0$ *then,*

$$\begin{aligned} \|q_1'\|^2 + \beta^2 \|q_1\|^2 &\leq C \beta^{-2} \|f\|^2\,, \\ \|q_2'\|^2 + \beta^2 \|q_2\|^2 &\leq C \|f\|^2\,, \end{aligned}$$

where $C > 0$ *depend only on a lower bound for* $L\beta$.

Estimation of α_j

Term 1: $i\omega(e_z \times f_t, F_{t1,j})$ contributing to r_1. Skipping factor $i\omega$, we have:

$$\begin{aligned}
\sum_j \int_0^L |\alpha_j|^2\, dz &\lesssim \sum_j \int_0^L \beta_j^{-2} |(e_z \times f_t, F_{t1,j} + \delta F_{t1,j})|^2 && \text{(Lemma 7 (ii)}_1\text{)}\\
&\lesssim 2 \sum_j \int_0^L [|(e_z \times f_t, F_{t1,j})|^2 + |(e_z \times f_t, \delta F_{t1,j})|^2] && \text{(Young's inequality)}\\
&\lesssim 2 \sum_j \int_0^L |(e_z \times f_t, F_{t1,j})|^2 && \text{(linearization)}\\
&\leq 2 \int_0^L \|e_z \times f_t\|^2\, dz\\
&= 2 \int_0^L \|f_t\|^2\, dz\,.
\end{aligned}$$

Term 2: $(\frac{1}{\epsilon} g_3, \operatorname{div} G_t)$ contributing to r_1.

$$\begin{aligned}
\sum_j \int_0^L |\alpha_j|^2\, dz &\lesssim \sum_j \int_0^L \beta_j^{-2} |(\tfrac{1}{\epsilon} g_3, \operatorname{div}(F_{t1,j} + \delta F_{t1,j}))|^2 && \text{(Lemma 7 (ii)}_1\text{)}\\
&\leq 2 \sum_j \int_0^L \beta_j^{-2} [|(\tfrac{1}{\epsilon} g_3, \operatorname{div}(F_{t1,j}))|^2 + |(\tfrac{1}{\epsilon} g_3, \operatorname{div}(\delta F_{t1,j}))|^2] && \text{(Young's lemma)}\\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-4} |(\tfrac{1}{\epsilon} g_3, \operatorname{div}(F_{t1,j}))|^2 && \text{(linearization)}\\
&\lesssim 0 && (\operatorname{div} F_{t1,j} = 0)
\end{aligned}$$

Estimation of α_j - Cont.

Term 3: $i\omega(e_z \times f_t, G_{t1,j})$ contributing to r_2. We follow exactly the same reasoning as for Term 1, sparing a factor β_j^{-2}.
Term 4: $(f_3, \operatorname{div} G_{t1,j})$ contributing to r_2.

$$
\begin{aligned}
\sum_j \int_0^L |\alpha_j|^2\, dz &\lesssim \sum_j \int_0^L \beta_j^{-2} |(f_3, \operatorname{div}(G_{t1,j} + \delta G_{t1,j}))|^2 && \text{(Lemma 7 (ii)}_1\text{)} \\
&\leq 2 \sum_j \int_0^L \beta_j^{-2} [|(f_3, \operatorname{div}(G_{t1,j}))|^2 + |(f_3, \operatorname{div}(\delta G_{t1,j}))|^2] && \text{(Young's lemma)} \\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-2} |(f_3, \operatorname{div}(G_{t1,j}))|^2 && \text{(linearization)} \\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-2} \mu_j |(f_3, \mu_j^{1/2} \psi_j)|^2 && (\beta_j^{-2}\mu_j \approx O(1)) \\
&\lesssim 2 \sum_j |(f_3, \mu_j^{1/2} \psi_j)|^2 = 2\|f_3\|^2 .
\end{aligned}
$$

Estimation of $\operatorname{curl} E_t$.
We need to estimate:

$$
\sum_i \int_0^L \underbrace{(\mu_i + \|\delta\epsilon\|_{L^\infty(D)})}_{\sim \beta_i^2} |\alpha_i|^2\, dz .
$$

We follow exactly the same strategy as above. In all cases, we can accommodate the extra β_i^2 factor.

Final Result

We follow the same reasoning for the remaining coefficients $\delta_j, \beta_j, \eta_j$ to arrive at our final result.

Theorem

Let $\Omega = D \times (0, L)$*. Assume that the dielectric constant* ϵ *is a sufficiently*[a] *small perturbation of a constant. There exists then a constant* $C > 0$*, independent of data* f, g *and solution* E, H *such that*

$$\|E\|^2_{L^2(\Omega)} + \|H\|^2_{L^2(\Omega)} \leq CL^2 \left(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)} \right) .$$

[a] So that the perturbation technique based on linearization is justified.

Thank you for your attention !

References

(1) M. Melenk, L. Demkowicz, and S. Henneking, ``Stability analysis for electromagnetic waveguides. Part 1: Acoustic and homogeneous electromagnetic waveguides.,'' Oden Institute, The University of Texas at Austin, Austin, TX 78712, Tech. Rep. 2, 2023.

(2) L. Demkowicz, M. Melenk, S. Henneking, and J. Badger, ``Stability analysis for acoustic and electromagnetic waveguides. Part 2: Non-homogeneous waveguides.,'' Oden Institute, The University of Texas at Austin, Austin, TX 78712, Tech. Rep. 3, 2023.