Using High-Order Element-based Galerkin Methods to capture Hurricane Intensification

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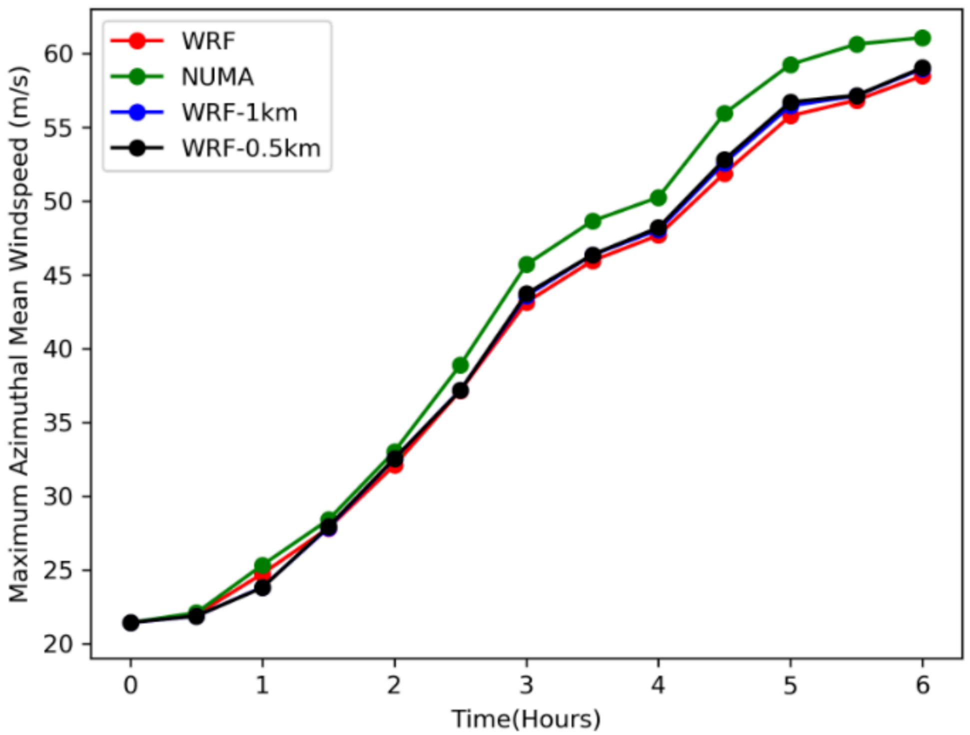
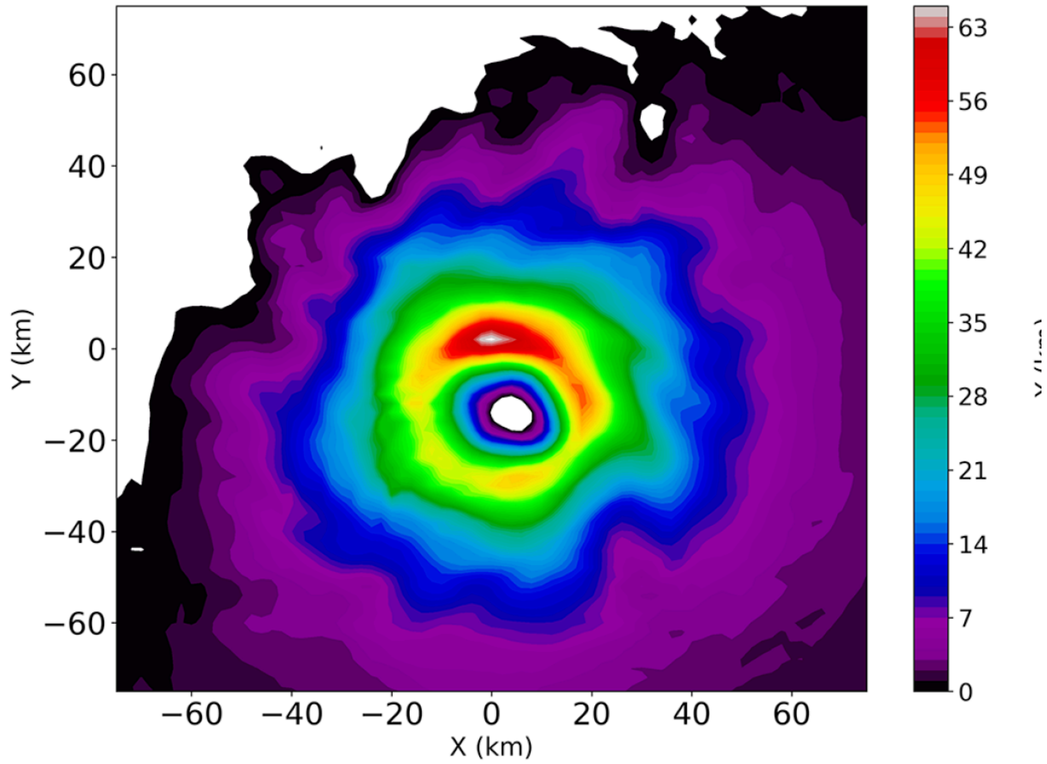
- U.S. Navy's NEPTUNE dynamics is based on NUMA. For more info see: https://frankgiraldo.wixsite.com/mysite/numa

- xNUMA is a light-weight version of NUMA designed specifically for the MMF + MR work presented here. More info can be found at: https://frankgiraldo.wixsite.com/mysite/xNUMA

Motivation

- Hurricane Rapid Intensification (RI) (where max wind velocities exceed ~30 mph within 24-hour period) continues to be an important topic to understand extreme weather.





- To properly capture RI requires LES models at $\Delta x = \mathcal{O}(100\,m)$ that run **stably** with as little **dissipation** as possible. Running a CRM at this resolution is still too expensive (cannot be done on a regular basis). Our simulations in [1] were run at $\Delta x = \mathcal{O}(2\,km)$ (CRM with 80 million DoF). LES would require 32 billion DoF with Δt 20x smaller (i.e., $\mathcal{O}(10^4)$ more expensive).

- To run such simulations require different strategies: to resolve fine-scale features, for time-integration, and in high-performance computing.

[1] Hasan et al., The effects of numerical dissipation on hurricane rapid intensification with observational heating, JAMES Vol 14 Issue 8 (2022)

Governing Equations

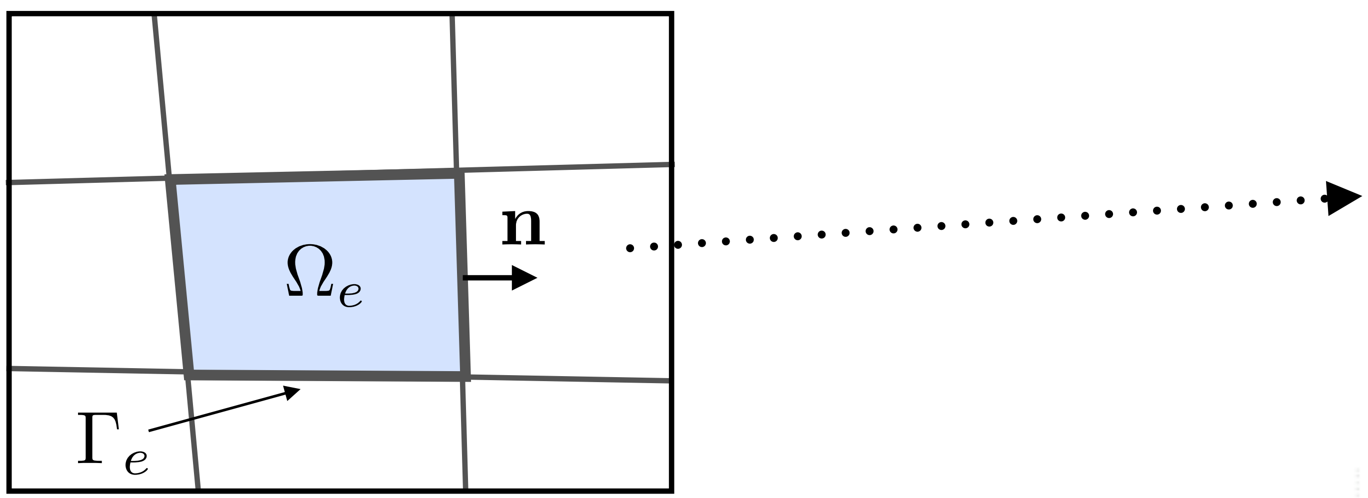
- Compressible Euler Equations (Non-hydrostatic)
  - $$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{u}\right) = 0,$$
  - $$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla P + \nabla \phi + \mathbf{\Omega} \times \mathbf{u} = S_P(\mathbf{u}),$$
  - $$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = S_P(\theta)$$
  - $$\frac{\partial q_i}{\partial t} + \mathbf{u} \cdot \nabla q_i = S_P(\mathbf{q_i}),\ i = 1, \ldots, M_{water}$$
- NUMA also carries an internal energy form (used for space weather applications) [2] and two conservation forms (used for Entropy-Stable work).

[2] Kelly et al., A physics-based open atmosphere boundary condition for heigh-coordinate atmospheric models, Journal of Computational Physics Vol 482 (2023)

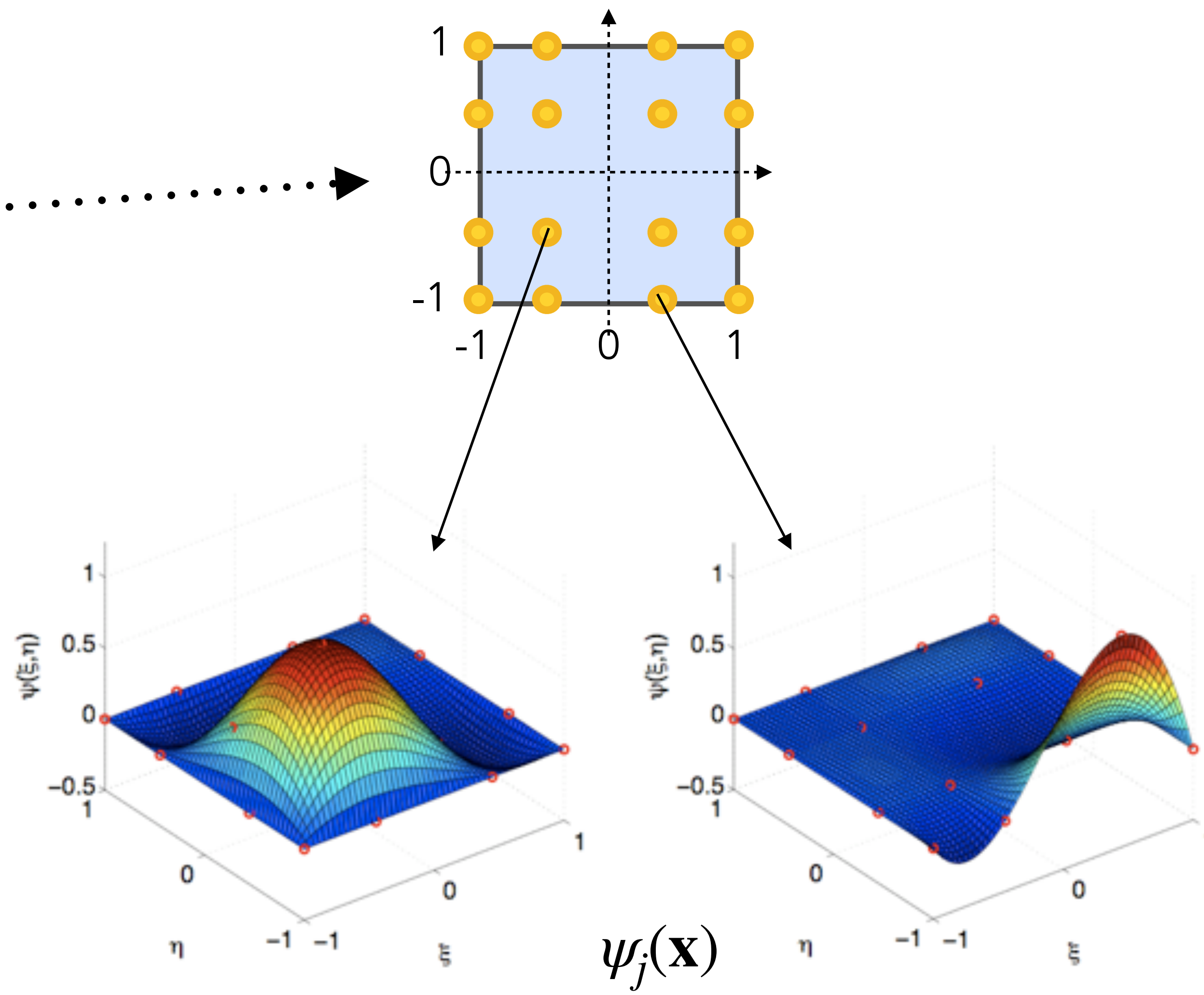
Element-based Galerkin Methods [3]

Domain decomposition

$$\Omega = \sum_{e=1}^{N_e} \Omega_e$$



Reference element



Basis functions - Lagrange polynomials

Approximate local solution as:

$$q_N^{(e)}(\mathbf{x}, t) = \sum_{j=1}^{M_N} \psi_j(\mathbf{x}) q_j^{(e)}(t)$$

[3] F.X. Giraldo. An Introduction to Element-Based Galerkin Methods on Tensor-Product Bases: Analysis, Algorithms, and Applications, Springer Texts in Computational Science and Engineering, Series Volume 24, ISBN: 978-3-030-55068-4 (2020)

Continuous/discontinuous Galerkin methods

Governing equation: $$\frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{q}) = S(\mathbf{q})$$

Approximate the **global** solution as: $$q_N^{(e)}(\mathbf{x}, t) = \sum_{j=1}^{M_N} \psi_j(\mathbf{x}) q_j^{(e)}(t)$$

$$\mathbf{F}_N = \mathbf{F}(\mathbf{q}_N)$$
$$S_N = S(\mathbf{q}_N)$$

Define residual: $$R\left(q_N^{(e)}\right) \equiv \frac{\partial \mathbf{q}_\mathbf{N}^{(\mathbf{e})}}{\partial t} + \nabla \cdot \mathbf{F}_N^{(e)} - S_N^{(e)} = \epsilon$$

Problem statement:

test function

Find $\mathbf{q}_N \in \mathcal{S} \quad \forall \psi \in \mathcal{S}$ $\begin{cases} \mathcal{S}_{CG} = \{\psi \in H^1(\Omega) : \psi \in P_N(\Omega_e)\ \forall \Omega_e\} \\ \mathcal{S}_{DG} = \{\psi \in L^2(\Omega) : \psi \in P_N(\Omega_e)\ \forall \Omega_e\} \end{cases}$

such that $$\int_{\Omega_e} \psi_i R(\mathbf{q}_N) \mathrm{d}\Omega_e = 0$$

Continuous/discontinuous Galerkin methods

Integral form:

$$\int_{\Omega_e} \psi_i R(\mathbf{q}_N^{(e)}) \mathrm{d}\Omega_e = 0$$

$$\int_{\Omega_e} \psi_i \frac{\partial \mathbf{q}_N^{(e)}}{\partial t} \mathrm{d}\Omega_e + \int_{\Omega_e} \psi_i \nabla \cdot \mathbf{F}_N^{(e)} \mathrm{d}\Omega_e - \int_{\Omega_e} \psi_i S_N^{(e)} \mathrm{d}\Omega_e = 0$$

Integration by parts: $\int_{\Omega_e} \psi_i \nabla \cdot \mathbf{F}_N^{(e)} \mathrm{d}\Omega_e = \int_{\Omega_e} \nabla \cdot (\psi_i \mathbf{F}_N^{(e)}) \mathrm{d}\Gamma_e - \int_{\Omega_e} \nabla \psi_i \cdot \mathbf{F}_N^{(e)} \mathrm{d}\Omega_e = \int_{\Gamma_e} \mathbf{n} \cdot \psi_i \mathbf{F}_N^{(e)} \mathrm{d}\Gamma_e - \int_{\Omega_e} \nabla \psi_i \cdot \mathbf{F}_N^{(e)} \mathrm{d}\Omega_e$

$$\boxed{\int_{\Omega_e} \psi_i \frac{\partial \mathbf{q}_N^{(e)}}{\partial t} \mathrm{d}\Omega_e + \underbrace{\int_{\Gamma_e} \mathbf{n} \cdot \psi_i \mathbf{F}_N^{(e)} \mathrm{d}\Gamma_e}_{\text{face integral}} - \underbrace{\int_{\Omega_e} \nabla \psi_i \cdot \mathbf{F}_N^{(e)} \mathrm{d}\Omega_e - \int_{\Omega_e} \psi_i S_N^{(e)} \mathrm{d}\Omega_e}_{\text{volume integrals}} = 0}$$

Matrix form:

$$M_{ij}^{(e)} \frac{\mathrm{d}\mathbf{q}_j^{(e)}}{\mathrm{d}t} + \sum_{f=1}^{N_{faces}} \left(\mathbf{M}_{ij}^{(e,f)}\right)^T \mathbf{F}_j^{(e,f,*)} - \left(\tilde{\mathbf{D}}_{ij}^{(e)}\right)^T \mathbf{F}_{ij}^{(e)} - S_i^{(e)} = 0$$

$M_{ij}^{(e)} = \int_{\Omega_e} \psi_i \psi_j \mathrm{d}\Omega_e$ — mass matrix

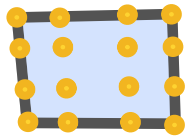
$\mathbf{M}_{ij}^{(e,f)} = \int_{\Gamma_e} \psi_i \psi_j \mathbf{n}^{(e,f)} \mathrm{d}\Gamma_e$ — face mass matrix

$\tilde{\mathbf{D}}_{ij}^{(e)} = \int_{\Omega_e} \nabla \psi_i \psi_j \mathrm{d}\Omega_e$ — differentiation matrix

Unified CG/DG methods

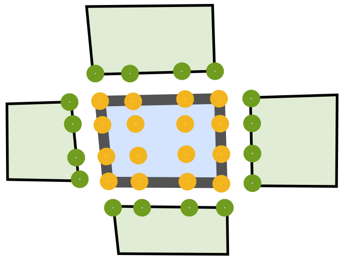
$$\int_{\Omega_e} \psi_i \frac{\partial \mathbf{q}_N^{(e)}}{\partial t}\, d\Omega_e + \int_{\Gamma_e} \psi_i \mathbf{n} \cdot \mathbf{F}_N^{(*)}\, d\Gamma_e - \int_{\Omega_e} \nabla \psi_i \cdot \mathbf{F}_N^{(e)}\, d\Omega_e - \int_{\Omega_e} \psi_i \mathbf{S}_N^{(e)}\, d\Omega_e = \mathbf{0}$$

1. Evaluate "volume" integrals on element interiors



$$R^{(e)} := \int_{\Omega_e} \nabla \psi_i \cdot F_N^{(e)}\, d\Omega_e + \int_{\Omega_e} \psi_i S_N^{(e)}\, d\Omega_e$$

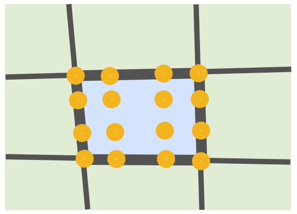
2. Evaluate flux integrals



$$R^{(e)} := R^{(e)} - \int_{\Gamma_e} \psi_i\, n \cdot F_N^{(*)}\, d\Gamma_e$$

CG: cancels at interior element edges

3. Direct Stiffness Summation



$$R = \bigwedge_{e=1}^{N_e} R^{(e)}$$

DG: matrices are block diagonal except for the flux matrix

4. Multiply by inverse global mass matrix and evolve time step

$$\frac{dq_i}{dt} = M_i^{-1} R_i$$

We rely on inexact integration so M is diagonal

- NUMA carries CG and DG, xNUMA CG, and ATUM DG

Flux-Difference DG methods

- Have also been exploring flux-difference DG methods
- First consider that discrete integration by parts (SBP property) is satisfied by Lobatto points with inexact integration

$$\left(D_{ij} + D_{ij}^T\right) f_j^{(e)} = F_{ij} f_j^{(e)} \rightarrow \int_{x_0}^{x_1} \left(\psi f_x + \psi_x f\right) dx = [\psi f]_{x_0}^{x_1}$$

- Such that row sum of D is zero:

$$\sum_{j=1}^{M_N} D_{ij} = 0$$

- This allows us to write Advection and Conservation forms in the same way (ignoring the boundary flux for simplicity)

$$M_i \frac{d\mathbf{q}_i^{(e)}}{dt} + D_{ij}\left(\mathbf{u}_k^{(e)} \mathbf{q}_j^{(e)}\right) = 0$$

- where k=j yields conservation form and k=i advection form

Flux-Difference DG methods

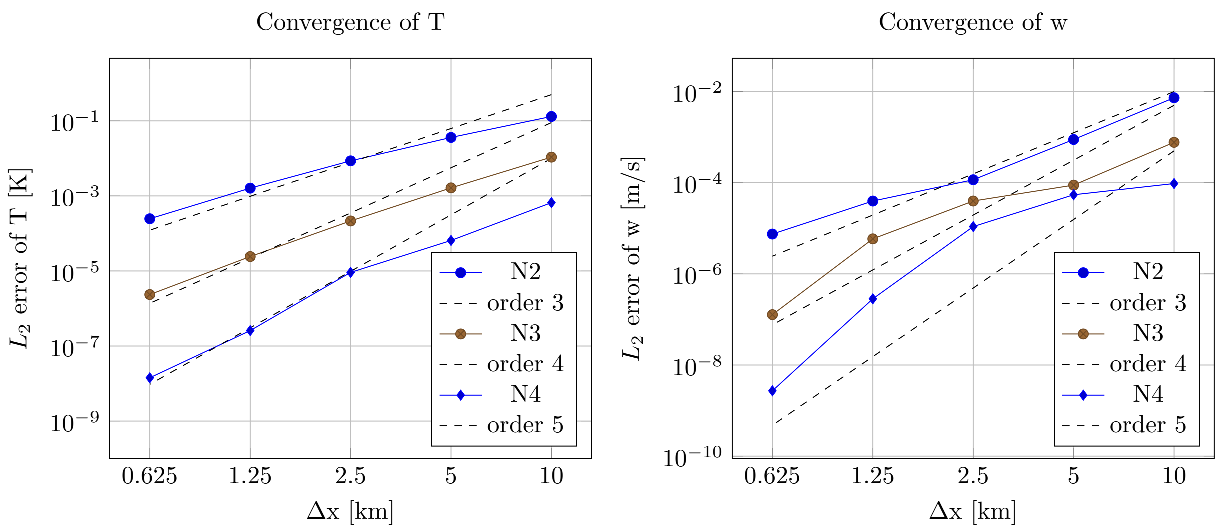
- For conservation laws discretized as:

$$M_i \frac{d\mathbf{q}_i^{(e)}}{dt} + D_{ij} f_j^{(e)} = \mathbf{0}$$

- We can write them in a general form as

$$M_i \frac{d\mathbf{q}_i^{(e)}}{dt} + \sum_{i=1}^{M_N} D_{ij} \circ \hat{f}_{ij}^{(e)} = \mathbf{0}$$

Inertia Gravity Wave with warped grid [5]



- with, e.g., the kinetic-energy-preserving flux [4] being

$$\hat{f}_{ij}^{(adv)}(q, \mathbf{u}) = 2[[\mathbf{u}_i]]\{\{q_j\}\}, \quad \hat{f}_{ij}^{(con)}(q, \mathbf{u}) = 2\{\{\mathbf{u}_i\}\}\{\{q_j\}\}$$

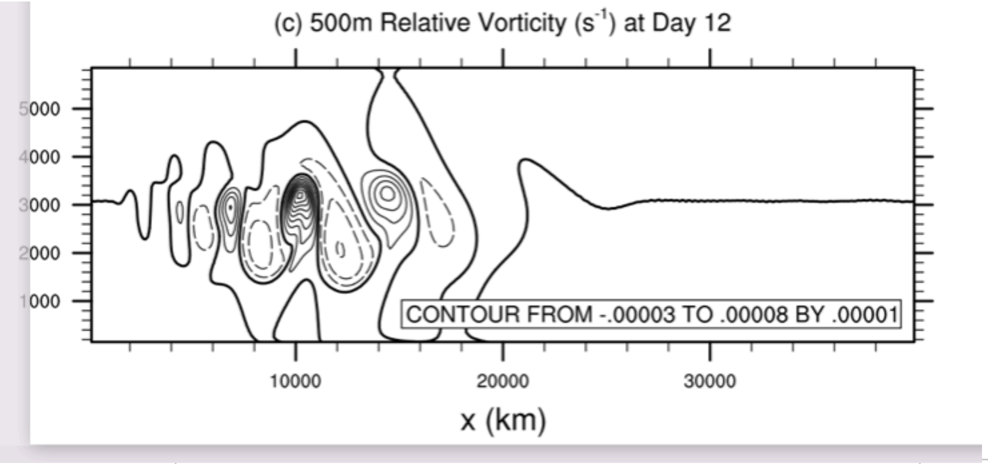
- and other options for entropy-stable flux [5]

- Let's see why this is important

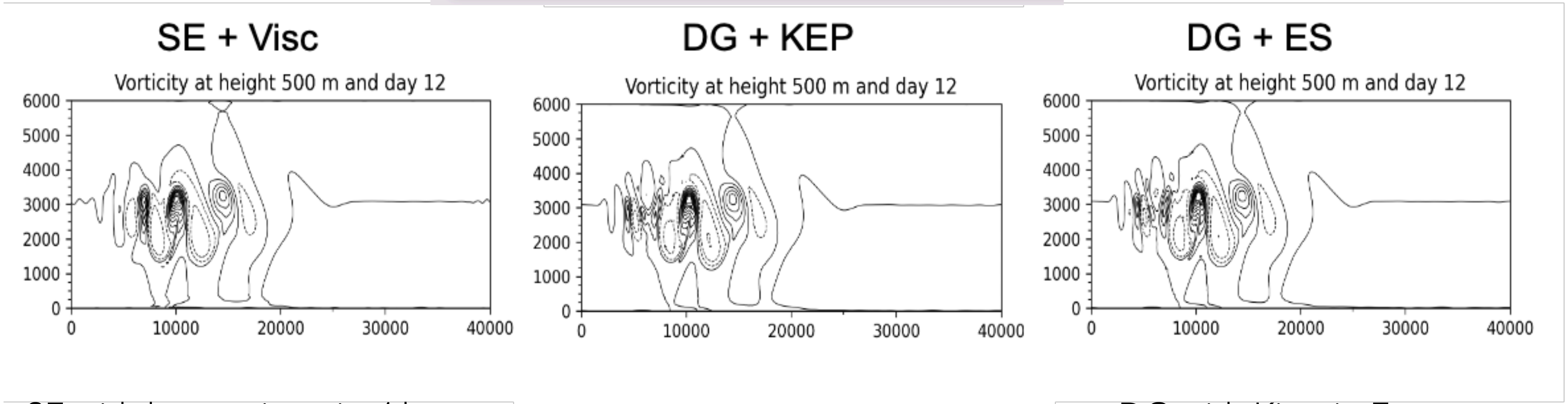
[4] Souza et al. The flux-differencing DG applied to an idealized fully compressible nonhydrostatic dry atmosphere, JAMES 2023
[5] Waruszewski et al. Entropy Stable DG for Balance Laws in Non-Conservative Form: Applications to the Euler Equations with Gravity JCP (2022)

Stabilizing EBG Methods

- Baroclinic Instability in a channel for 12-days using Grid Resolution: 100 km x 100 km x 1 km



WRF



- SE with hyper-viscosity (that needs to be tuned) yields similar results to DG with Kinetic-Energy-Preserving and Entropy-Stable flux (with no need to tune). All NUMA simulations are stable indefinitely.

- Conclusion: SE + Visc and DG KEP and ES offer stability with limited dissipation (good for hurricane RI); hyper-viscosity needs to be tuned. SE + Visc currently offers faster time-to-solution.

Multi-scale Methods

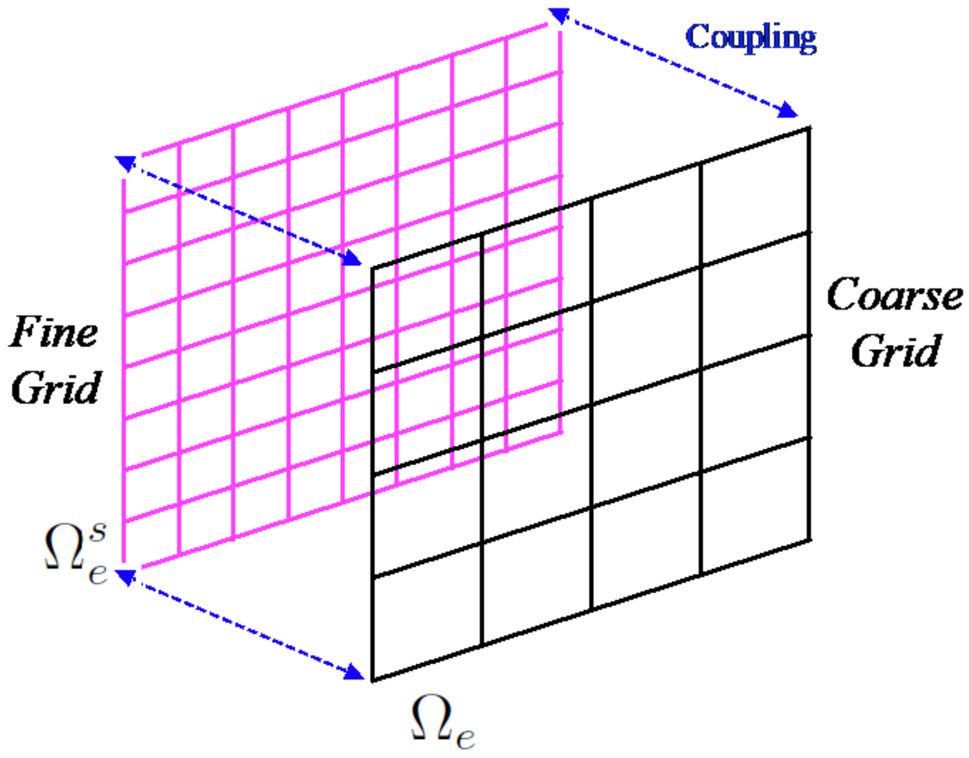
- Recalling multi-scale methods (see [6]) we consider the following PDE
  - $\dfrac{\partial q}{\partial t} = S(q)$ where $q = \bar{q} + \tilde{q}$ with $(q, \bar{q}, \tilde{q})$ represent the total, coarse-scale, and fine-scale solutions; $S(q)$ the RHS operator

- Using a VMS approach, we can decouple these (after some algebra) as

$$\int_{\Omega_e} \bar{\psi}_i \frac{\partial \bar{q}}{\partial t} d\Omega_e = \int_{\Omega_e} \bar{\psi}_i \left(S(\bar{q}) - \frac{\partial \tilde{q}}{\partial t} + \frac{\partial S}{\partial q}(\bar{q})\tilde{q} \right) d\Omega_e$$

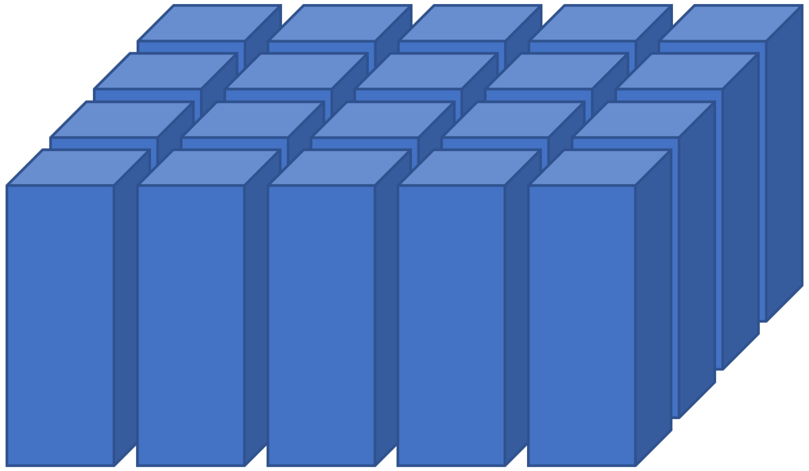
$$\int_{\Omega_e^s} \tilde{\psi}_i \frac{\partial \tilde{q}}{\partial t} d\Omega_e^s = \int_{\Omega_e^s} \tilde{\psi}_i \left(S(\bar{q}) - \frac{\partial \bar{q}}{\partial t} + \frac{\partial S}{\partial q}(\bar{q})\tilde{q} \right) d\Omega_e^s$$

- Unlike classical VMS, here we computationally resolve coarse and fine scale solutions. However, we are currently first considering a simpler approach.

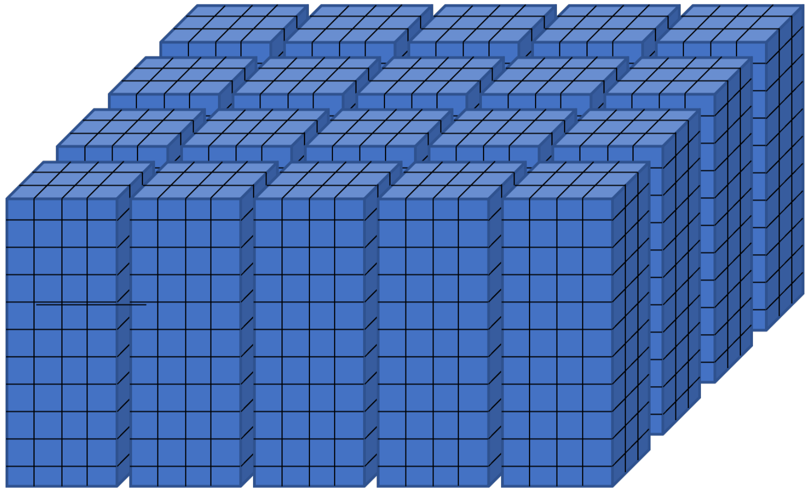


Multi-scale Methods

- Using the multi-scale modeling framework (MMF, see [7]) we consider the two PDEs
  - $\frac{\partial Q}{\partial T} = S(Q) + F(Q,q)$ and $\frac{\partial q}{\partial t} = s(q) + f(q,Q)$ where (Q,q) represent the coarse-scale and fine-scale solutions, (S,s) the RHS operators, and (F,f) the coupling (forcing) between the coarse- and fine-scales
- Using a standard super-parameterization approach, we solve the two problems denoted as



Q: Coarse-Scale Model:



q: Fine-Scale Model:

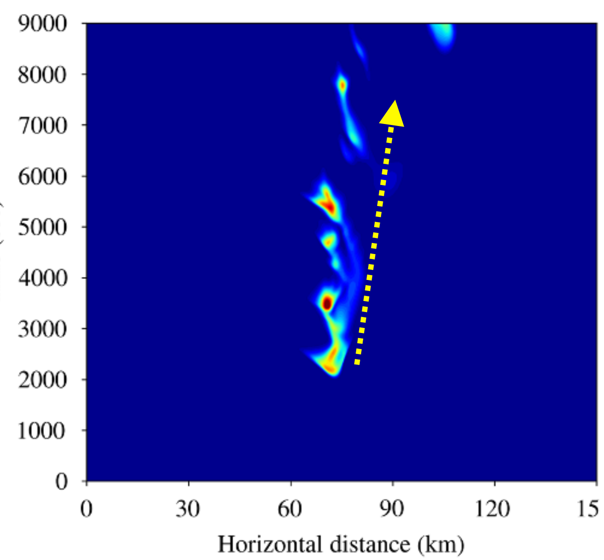
- Computational savings arises from elements only communicating at the Coarse-Scale model and at its larger time-step $\Delta T = m\,\Delta t$

[7] Grabowski and Smolarkiewicz, CRCP: a Cloud Resolving Convection Parameterization for modeling the tropical convecting atmosphere Physica D 1999

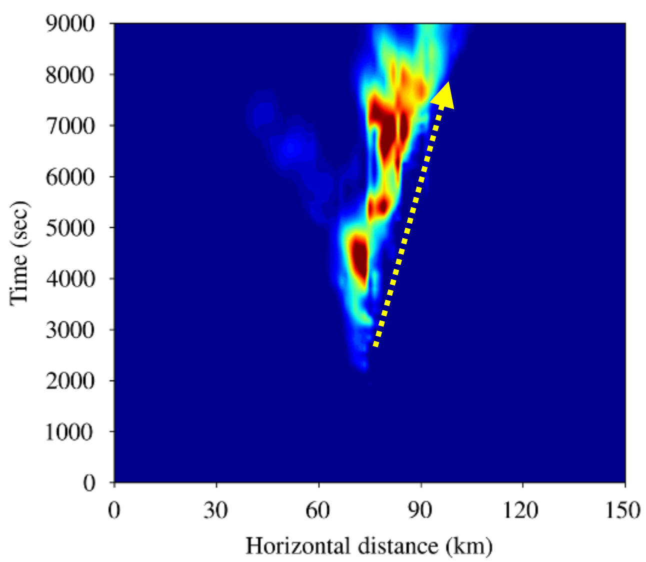
MMF Result: Squall Line

Time: 0 sec

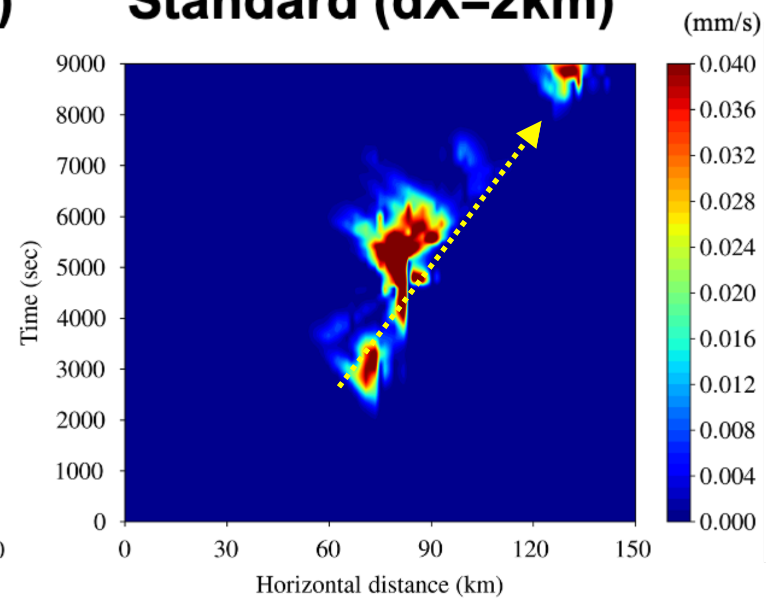
Fine Simulation
Standard (dX=200m)

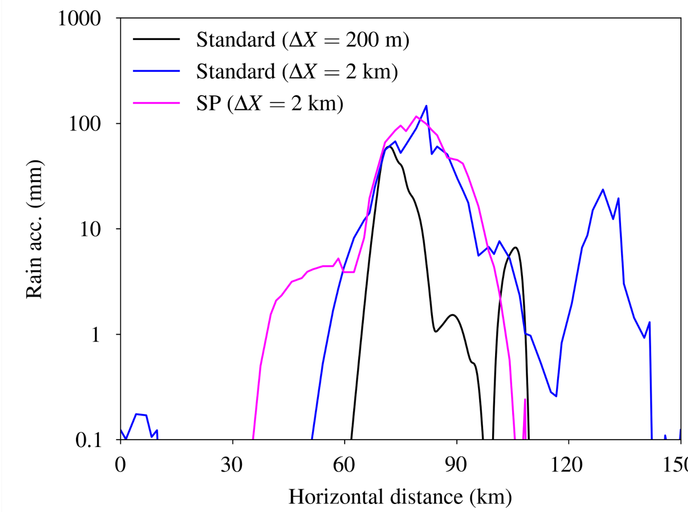


Superparameterization
SP (dX=2km/ dx=100m)



Coarse Simulation
Standard (dX=2km)





Work by Soonpil Kang (NPS)

Time-Integration

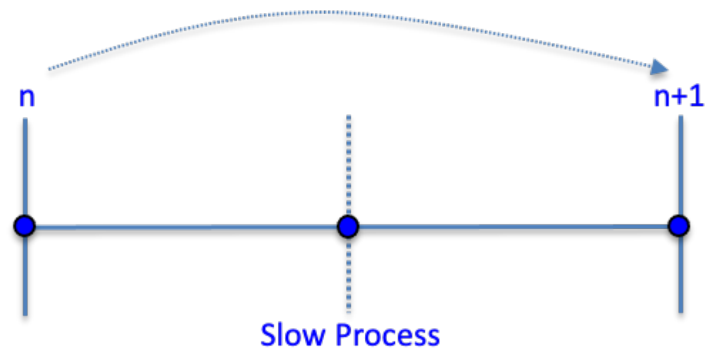
- The classical way to evolve CRM and LES in time are IMEX methods as follows:
  - $\frac{dq}{dt} = \left\{S(q) - L(q)\right\}_{EX} + [L(q)]_{IM} \equiv \mathcal{S}(q) + \mathcal{F}(q)$, where L(q) is a linear operator that extracts the acoustic waves and allows us to partition the ODE into its slow and fast components $(\mathcal{S}, \mathcal{F})$
  - We have relied on s-stage Additive Runge-Kutta (SDIRK) methods:
    - $Q^{(i)} = q^n + \Delta t \sum_{j=1}^{i-1} a_{ij}^E \mathcal{S}\left(Q^{(j)}\right) + \sum_{j=1}^{i} a_{ij}^I \mathcal{F}\left(Q^{(j)}\right),$
    - $q^{n+1} = q^n + \Delta t \sum_{j=1}^{s} b_i S\left(Q^{(j)}\right)$
  - Replacing $\mathcal{F} \rightarrow \mathcal{N}$ where $\mathcal{N}$ is a nonlinear operator is also possible but requires special care in solving (e.g., Newton's method, JFNK, etc.).
- For many problems, standard IMEX may not be optimal. In particular, if the slow process is expensive to compute and the fast process inexpensive.

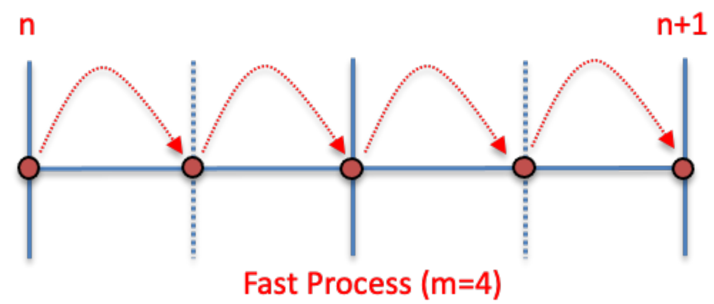
MMF + MR

- To understand why, let us first describe the current strategy used to evolve the MMF scheme:

1. $Q^{n+1} = Q^n + \Delta t \left[R(Q^{n+1}, q^n)\right],$

2. $q^{n+\frac{i}{m}} = q^{n+\frac{(i-1)}{m}} + \dfrac{\Delta t}{m} \left[R(q^{n+\frac{i}{m}}, Q^{n+1})\right], i = 1,\ldots,m$





  - Here we use IMEX-ARK1 but the coupling terms are handled the same even for, say, IMEX-ARK2 or higher, where coupling error is $\mathcal{O}(\Delta t)$.

  - Can we do better?

MMF + MR

- We propose a more consistent coupling using multirate (MR) (e.g., see [8]) as follows, where the stages are **synchronized**. The stage values are given as

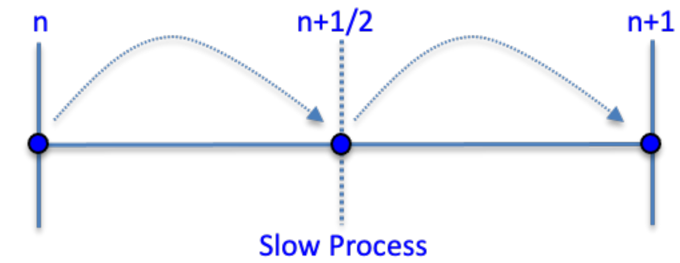
1. $Q^{(j^O)} = Q^n + \Delta t \sum_{k=1}^{j^O-1} a^O_{j^O,k} R\left(Q^{(j^O)}, q^{(*)}\right),$

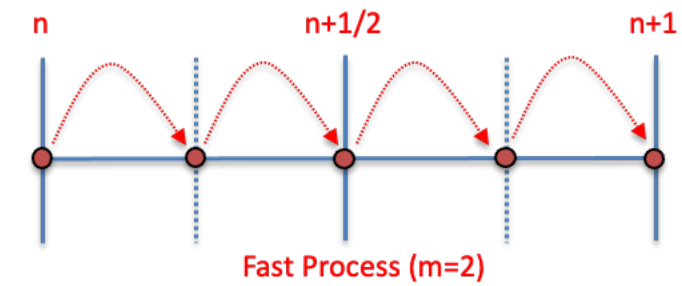
2. $q^{(i^I,j^O)} = q^{(1,j^O)} + \left(c^O_{j^O} - c^O_{j^O-1}\right) \Delta t \sum_{k=1}^{i^I-1} a^I_{i^I,k} R\left(q^{(k,j^O)}, Q^{(*)}\right),$

3. $q^{(j^O)} = q^{(1,j^O)} + \tilde{c}^O_{j^O} \Delta t \sum_{i=1}^{s^I} b^I_i R\left(q^{(i,j^O)}, Q^{(*)}\right),$

4. $Q^{(j^O)} = Q^n + \Delta t \sum_{i=1}^{s^O} b^O_i R\left(Q^{(i)}, q^{(i)}\right)$

- Some concerns: MIS [8] executes MR across stages and is general tableau indicates a likely issue with conservation of linear invariants MPRK [9] allows for conservation but limited to the same fast/slow method. This approach is certainly better than the standard approach but will it be too expensive?





$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array} \qquad \begin{array}{c|ccc} 0 & 0 & & \\ c_2 & a_{2,1} & & \\ c_3 & a_{3,1} & a_{3,2} & \\ \hline & b_1 & b_2 & b_3 \end{array}$$

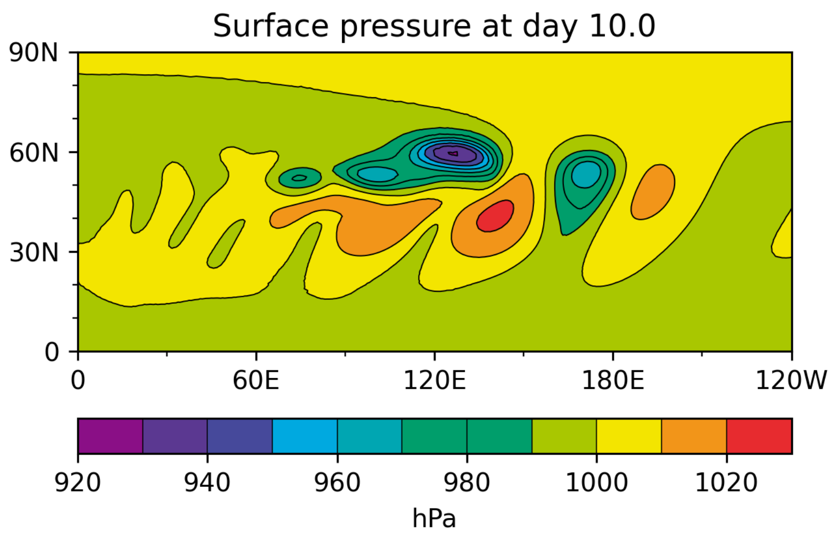
$$\begin{array}{c|cccccc}
c^{(1,2)} & 0 & & & & & \\
c^{(2,2)} & \tilde{c}^O_2 a^I_{2,1} & & & & & \\
c^{(3,2)} & \tilde{c}^O_2 a^I_{3,1} & \tilde{c}^O_2 a^I_{3,2} & & & & \\
\hdashline
c^{(1,3)} = \tilde{c}^O_2 & \tilde{c}^O_2 b^I_1 & \tilde{c}^O_2 b^I_2 & \tilde{c}^O_2 b^I_3 & & & \\
\hdashline
c^{(2,3)} & \tilde{c}^O_2 b^I_1 & \tilde{c}^O_2 b^I_2 & \tilde{c}^O_2 b^I_3 & \tilde{c}^O_3 a^I_{2,1} & & \\
c^{(3,3)} & \tilde{c}^O_2 b^I_1 & \tilde{c}^O_2 b^I_2 & \tilde{c}^O_2 b^I_3 & \tilde{c}^O_3 a^I_{3,1} & \tilde{c}^O_3 a^I_{3,2} & \\
\hline
\tilde{c}^O_2 + \tilde{c}^O_3 = 1 & \tilde{c}^O_2 b^I_1 & \tilde{c}^O_2 b^I_2 & \tilde{c}^O_2 b^I_3 & \tilde{c}^O_3 b^I_1 & \tilde{c}^O_3 b^I_2 & \tilde{c}^O_3 b^I_3
\end{array}$$

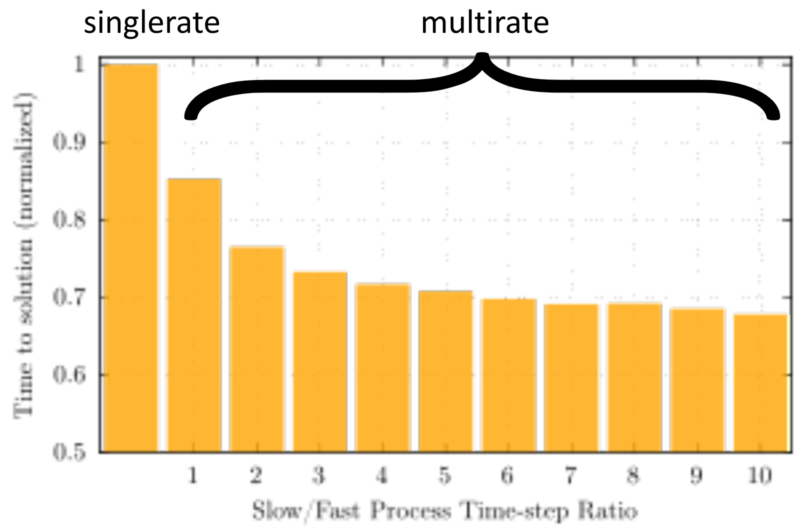
3-stage 2-Rate Fast Tableau

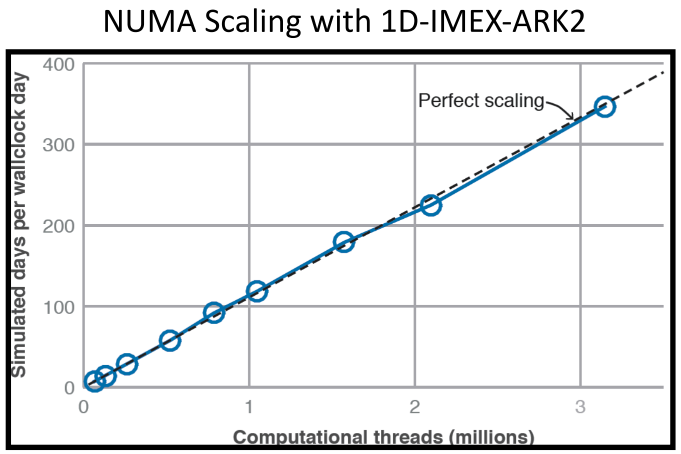
[8] Schlegel, Knoth, Arnold, Wolke, Multirate RungeKutta schemes for advection equations, JCAM 2009
[9] Constantinescu, Sandu, Multirate Timestepping Methods for Hyperbolic Conservation Laws, JSC 2007

Multirate Time-Integration

- Let us consider the baroclinic instability on the 3-sphere using ARK1 for hyper-diffusion and 1D-IMEX ARK2 for the remaining operators.







- Multirate yields 30% increase in performance over singlerate 1D-IMEX-ARK2 (which we know already scales well). We have yet to exploit all possibilities offered in:

1. Patrick Mugg, Extrapolated Multirate Methods for Hyperbolic Partial Differential Equations, NPS PhD Thesis (June 2021); paper in preparation

2. Alves, Kelly, Giraldo, Implicit time-integrators for global nonhydrostatic atmospheric modeling (in preparation)

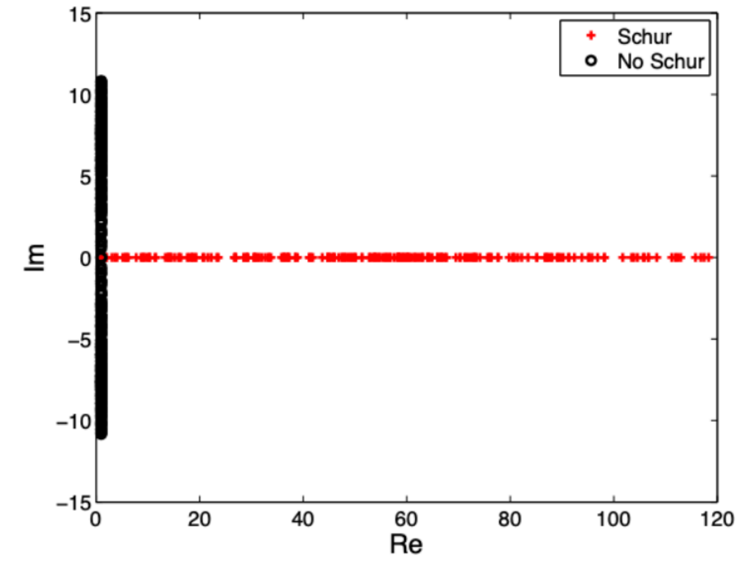
IMEX: No-Schur vs Schur Form

- For the shallow water equations:

$$\phi_t + \nabla \cdot \mathbf{U} = 0$$
$$\mathbf{U}_t + \nabla \phi = 0$$

- Discretizing in both space (via strong form CG) and time (assume ARK1) results in the No-Schur form:

$$M\phi^{n+1} + \Delta t\, \mathbf{D}^T \mathbf{U}^{n+1} = M\phi^n$$
$$M\mathbf{U}^{n+1} + \Delta t\, \mathbf{D}\phi^{n+1} = M\mathbf{U}^n$$



- Applying a block LU factorization (or subbing $\mathbf{U}^{n+1}$ into ϕ^{n+1} equation) yields the Schur form:

$$M\phi^{n+1} - \Delta t^2\, \mathbf{D}^T M^{-1} \mathbf{D}\, \phi^{n+1} = R^n$$

- A few observations: 1) we need M^{-1}; 2) This system is smaller ($\mathcal{O}(N^2)$ instead of $\mathcal{O}(N_{var}N)^2$); 3) better conditioned [10].

10] F.X. Giraldo, M. Restelli, and M. Laeuter, Semi-implicit formulations of the Euler equations: applications to nonhydrostatic atmospheric modeling, SIAM J. Sci. Comp., Vol. 32, 3394-3425 (2010)

IMEX: No-Schur vs Schur Form

- For general systems of equations we write (after integration by parts):

$$\int_{\Omega_e} \psi_i \frac{\partial \mathbf{q}^{(e)}}{\partial t} d\Omega_e + \int_{\Gamma_e} \psi_i \mathbf{n} \cdot \mathbf{F}^{(*)} d\Gamma_e - \int_{\Omega_e} \nabla \psi_i \cdot \mathbf{F}^{(e)} d\Gamma_e = \int_{\Omega_e} \psi_i \mathbf{S}^{(e)} d\Omega_e$$

- Writing the numerical fluxes as:

$$\mathbf{F}^{(*)} = \{\{\mathbf{F}\}\} - \frac{\lambda \mathbf{n}}{2}[[\mathbf{q}]] \longrightarrow \int_{\Gamma_e} \psi_i \mathbf{n} \cdot \mathbf{F}^{(*)} d\Gamma_e = \mathbf{C}_G^T \mathbf{F} - \mathbf{J}_G^T \mathbf{q}$$

- Discretizing in space (via EBG) yields: $M \frac{d\mathbf{q}}{dt} + \mathbf{C}^T \mathbf{F} - \mathbf{J}^T \mathbf{q} - \mathbf{D}^T \mathbf{F} = S$

- Discretizing in time (via ARK1) yields: $(M - \Delta t \mathbf{J})\, \mathbf{q}^{n+1} + \Delta t [\mathbf{C}^T \mathbf{F} - \mathbf{D}^T \mathbf{F} - S]^{n+1} = M \mathbf{q}^n$

If J non-empty then no longer block diagonal so computing $(M - \Delta t \mathbf{J})^{-1}$ is non-trivial

- **Why this matters**: To construct the Schur complement, we need to isolate terms from one equation to substitute into another. To do this, we need to invert the mass matrix

IMEX: No-Schur vs Schur Form

- For DG we can avoid this difficulty by separating **Linear** and **Nonlinear** fluxes:

$$\int_{\Omega_e} \psi_i \frac{\partial \mathbf{q}^{(e)}}{\partial t} d\Omega_e + \int_{\Gamma_e} \psi_i \mathbf{n} \cdot (\mathbf{F}_L + \mathbf{F}_{NL})^{(*)} d\Gamma_e - \int_{\Omega_e} \nabla \psi_i \cdot (\mathbf{F}_L + \mathbf{F}_{NL})^{(e)} d\Gamma_e = \int_{\Omega_e} \psi_i \mathbf{S}^{(e)} d\Omega_e$$

- Which we write in matrix-vector form:

$$\left(M - \Delta t \mathbf{J}_L\right) \mathbf{q}^{n+1} + \Delta t \left(\mathbf{C}^T - \mathbf{D}^T\right) \mathbf{F}_L^{n+1} = \left(M + \Delta t \mathbf{J}_{NL}\right) \mathbf{q}^n - \Delta t [S + \left(\mathbf{C}^T - \mathbf{D}^T\right) \mathbf{F}_{NL}]^n$$

- Flux Choices: If $\mathbf{J}_L = 0$ then we can easily isolate $\mathbf{q}^{n+1}$

- With the proper selection of fluxes [11] we can construct a stable DG Schur form.

[11] S. Reddy, M. Waruszewski, F.A.V. B. Alves, F.X. Giraldo, IMplicit-EXplicit formulations for discontinuous Galerkin non-hydrostatic atmospheric models, JCP (in revision, May 2023)

IMEX: DG Schur Form

- No-Schur Form: $\rho_{tt} = \hat{\rho} - \alpha \nabla \cdot \mathbf{U}_{tt}, \qquad \mathbf{U}_{tt} = \hat{\mathbf{U}} - \alpha \left(\nabla P_{tt} + \rho_{tt} \nabla \phi \right)$

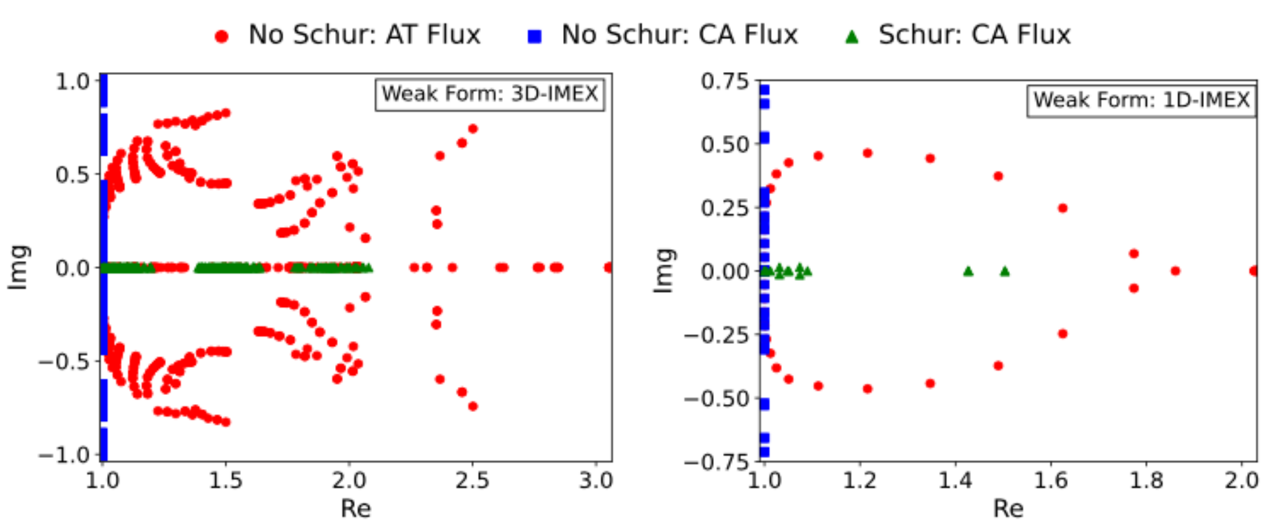
$$E_{tt} = \hat{E} - \alpha \nabla \cdot \left(h_0 \mathbf{U}_{tt} \right), \qquad P_{tt} = (\gamma - 1)(E_{tt} - \rho_{tt}\phi)$$

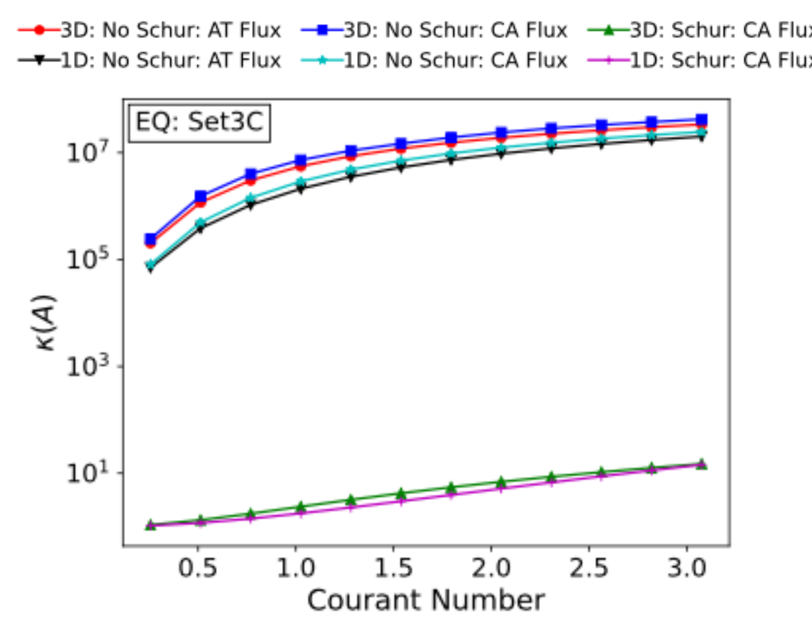
- Schur Form for Pressure (plugging $\rho_{tt} \rightarrow \mathbf{U}_{tt} \rightarrow E_{tt}$):

$$P_{tt} - \alpha^2(\gamma - 1)[\nabla \cdot (h_0 \mathbf{L}) - \phi \mathbf{L}] = (\gamma - 1)[\hat{E} - \phi\hat{\rho} - \alpha(\nabla \cdot (h_0 \mathbf{R}) - \phi \nabla \cdot \mathbf{R})]$$

$$\mathbf{L} = A^{-1}[\nabla P_{tt} + \frac{G}{\gamma - 1} P_{tt})], \qquad \mathbf{R} = A^{-1}[\hat{\mathbf{U}} - \alpha \mathbf{G}(h_0\hat{\rho} - \hat{E})] \qquad A = \mathbf{I} + \alpha^2 \mathbf{G} \nabla h_0$$

- Properties of Schur Form: all eigenvalues are real and positive (SPD as expected for a Helmholtz operator), see [11].

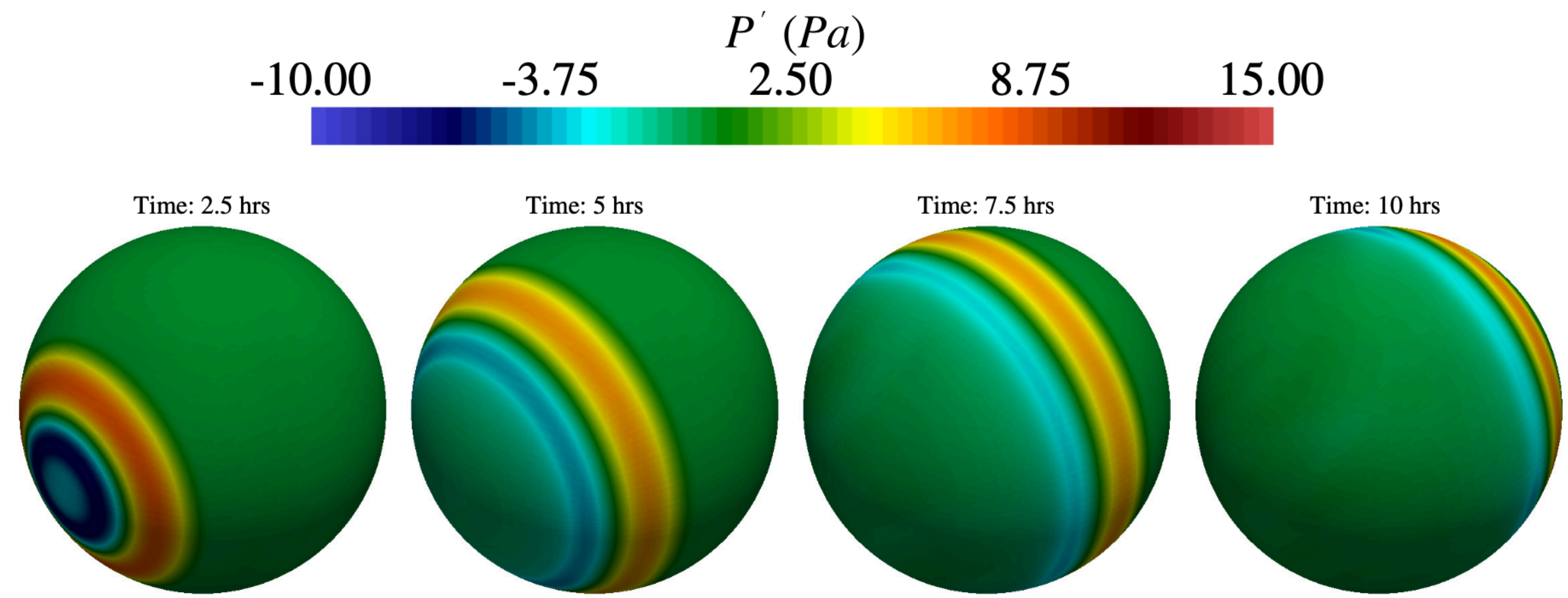


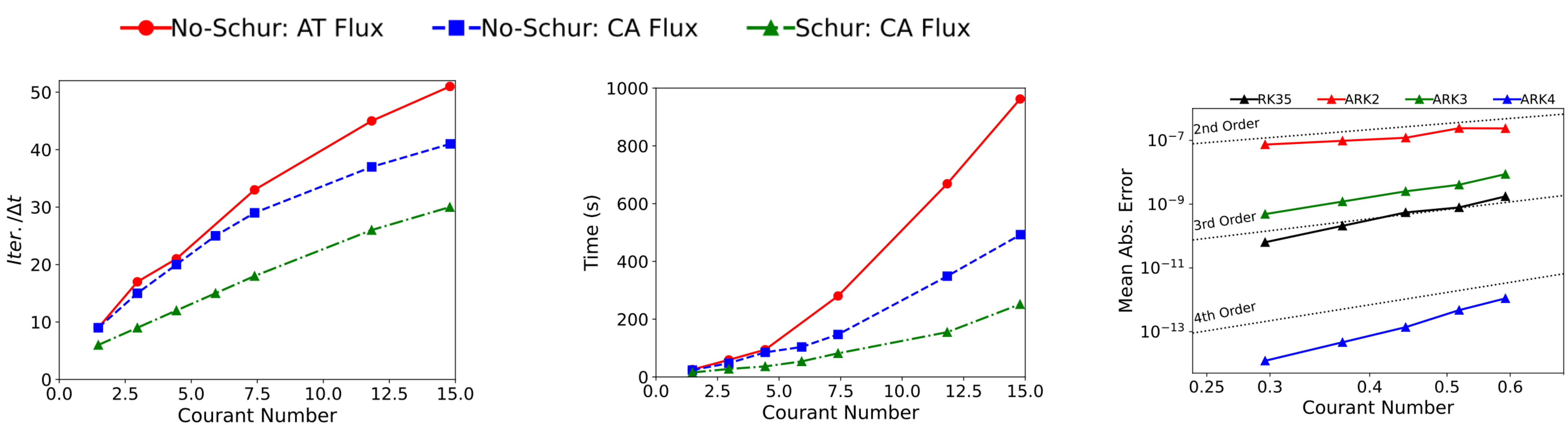


[11] S. Reddy, M. Waruszewski, F.A.V. B. Alves, F.X. Giraldo, IMplicit-EXplicit formulations for discontinuous Galerkin non-hydrostatic atmospheric models, JCP (in revision, May 2023)

IMEX: DG Schur Form

Acoustic Wave on the3-Sphere:





Iterations **Time-to-Solution** **Convergence Rates**

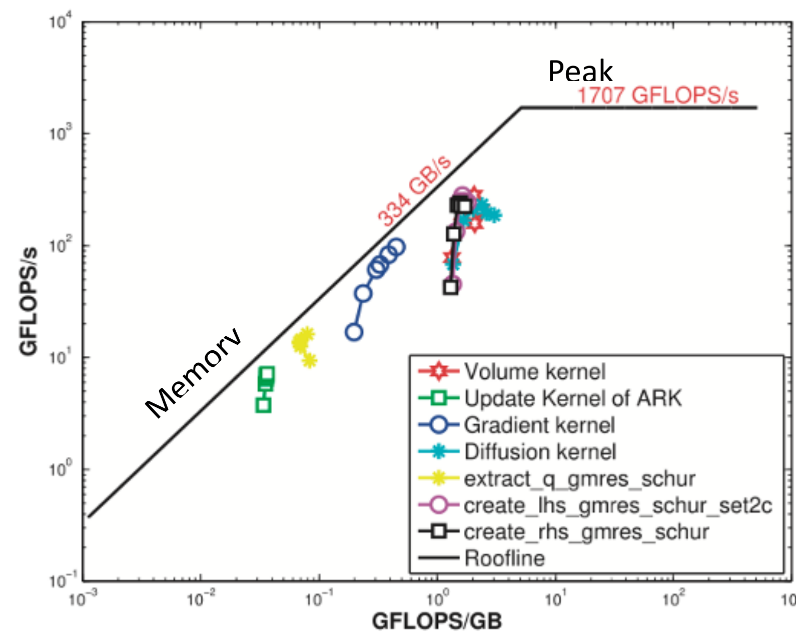
Summary

- EBG offers elegant way to approximate spatial derivatives of PDEs.
    - EBG methods offer much flexibility: geometric and algorithmic.
    - Schur forms offer an increase in performance as stand-alone methods or as preconditioners for more sophisticated methods (e.g., nonlinear IMEX).
- Flux-Differencing perform well under the stability metric but are computationally expensive (ES is 10x more expensive than hyper-diffusion and KEP is 5x due to a variety of reasons such as the time-step restriction and number of FLOPS).
- Combing Flux-Differencing DG with Schur forms is appealing.

Summary

- Our MMF model (xNUMA) is able to launch one CRM with an arbitrary number of LES models in both 2D and 3D.
  - Completed work: squall line test with warm rain [12].
  - Future work: Hurricane simulations with moisture (benchmark simulations in progress).
- Currently, CRM uses MPI. N LES runs:
  - use N MPI ranks to exploit hardware peak performance.
  - will use GPUs (work in progress).
  - Multirate time-integration offers a high-order, consistent, elegant, and powerful way to solve MMF problems.

From [13]



[12] S. Gabersek, F.X. Giraldo, and J. Doyle, Dry and moist idealized experiments with a two-dimensional spectral element model, MWR 2012
[13] D.S. Abdi, F.X. Giraldo, E.M. Constantinescu, L.E. Carr III, L. Wilcox, T. Warburton, Acceleration of the Implicit-Explicit NUMA on Manycore Processors, IJHPCA (2017)