An Efficient and Effective FEM Solver for Diffusion Equation with Strong Anisotropy

D. Green, X. Hu, J. Lore, L. Mu, and M. Stowell

Department of Mathematics
University of Georgia, Athens

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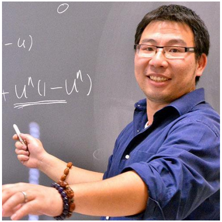
Virtual Seminar

Collaborators

- David Green, Oak Ridge National Laboratory
- Xiaozhe Hu, Tufts University
- Jeremy Lore, Oak Ridge National Laboratory
- Mark L. Stowell, Lawrence Livermore National Laboratory

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Outline

Figure: Major Fusion Breakthrough @ LLNL

- https://www.llnl.gov/news/national-ignition-facility-achieves-fusion-ignition

Figure: Unlimited Energy and Amazing Machine: Video from ITER

- References (Iter Website)
- https://www.iter.org/

Motivation and Background

- Magnetically confined plasma applications for which anisotropy is generated by the magnetic field.
- The anisotropy ratio between $D_{\parallel}/D_{\perp}$ can range from 10^6 (boundary) to 10^{12} (core region).
- Computational Challenges:
  - **Numerical Pollution** in the perpendicular diffusion or transport, due to the fast parallel dynamics

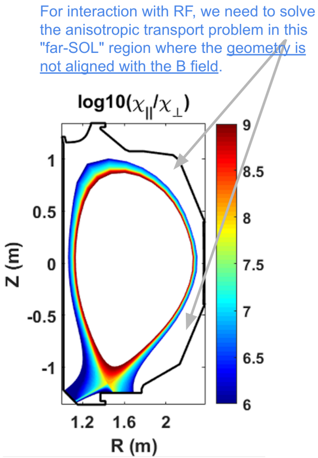

Figure: Demonstration of Anisotropy

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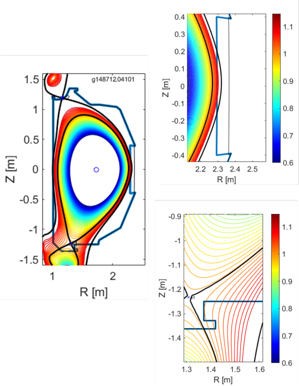

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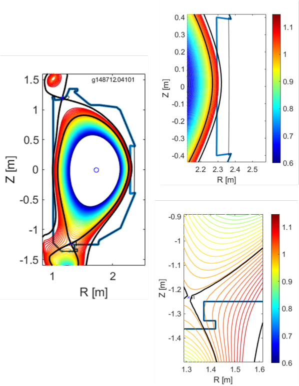
  - **For our far-SOL use case where we need non-field aligned coordinates to handle the high geometric fidelity in the boundary**



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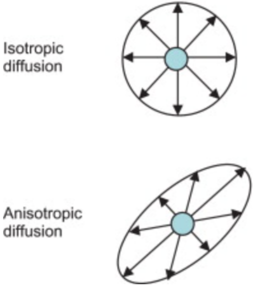
Literature Review: Spatial Discretization



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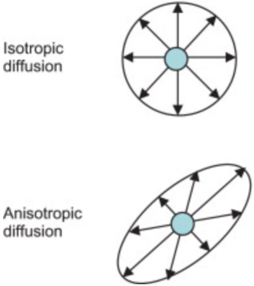

- Aligned Mesh
  - FDM with flux algined coordinates, [Dudson 2009, van Es 2014]
  - Finite Volume Method [Crouseilles 2015]
  - FEM with anisotropy adaptive mesh [Li 2010]
- Non-aligned Mesh
  - FDM with modified interpolation schemes [Soler2020]
  - FVM with high-order scheme [Holleman 2013]
  - **FEM with high-order scheme** [Gunter 2007, Held 2016, Giorgiani 2020]
  - Spectral element [Meier 2010]

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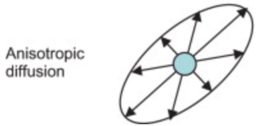

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  - Spectral element [Meier 2010]
- Fast Solver
  - Schwarz [Antonietti 2007], Multilevel [Dobrev 2006], Multigrid [Brenner 2005] Methods
  - Subspace Correction [Xu 1992] and **Auxiliary Space Preconditioning** [S. Nepomnyaschikh 1992, Xu 1996]

Goal of this project

We will address the issues in the Diffusion Equations with **Strong Anisotropy** ($D_{\|}/D_{\perp} \geq$1E6):

- Accuracy on the **Non-aligned Mesh**
- Efficiency of the **Linear Solver**

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- Efficiency of the **Linear Solver**
  - By Auxiliary Space Pre-conditioner

Problem Setting

We consider the following steady state anisotropic diffusion equation:

$$-\nabla \cdot (\mathbb{D}\nabla u) = f, \text{ in } \Omega, \quad (1)$$

$$u = 0, \text{ on } \partial\Omega, \quad (2)$$

where the diffusion coefficient tensor is given by

$$\mathbb{D} = \begin{pmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{pmatrix} \begin{pmatrix} D_{\parallel} & 0 \\ 0 & D_{\perp} \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{pmatrix}.$$

The direction of the anisotropy, or the magnetic field, is given by a unit vector $\mathbf{b} = (b_1, b_2)^\top$. Here $D_{\parallel}$ and $D_{\perp}$ represent the parallel and the perpendicular diffusion coefficient. For example, $D_{\perp} = 1$.

Finite Element Space

Let finite element spaces be

$$V_{\text{CG}} = \{v \in H_0^1(\Omega) |\ v|_T \in P_k(T), \forall T \in \mathcal{T}_h\}, \quad (3)$$

$$V_{\text{DG}} = \{v \in L^2(\Omega) |\ v|_T \in P_k(T), \forall T \in \mathcal{T}_h\}. \quad (4)$$

where $P_k(T)$ $(k \geq 1)$ denotes the polynomials with degree $\leq k$.

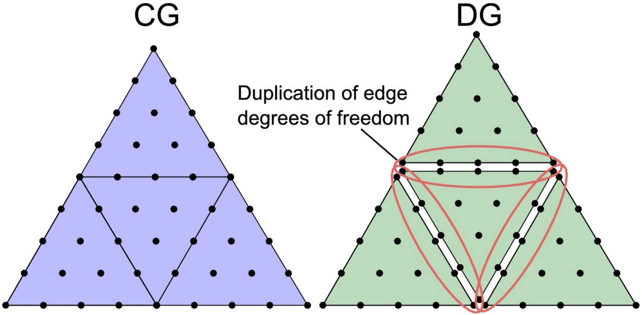

Figure: DG has more DoFs compared to CG on the same mesh.

Continuous Galerkin FEMs

The H^1-FEM is to find the numerical solution $u_h \in V_{\mathsf{CG}}$, such that

$$A_{\mathsf{CG}}(u_h, v) = \sum_{T \in \mathcal{T}_h} \int_T f v dT, \ \forall v \in V_{\mathsf{CG}}, \tag{5}$$

where the bilinear form

$$A_{\mathsf{CG}}(w, v) = \sum_{T \in \mathcal{T}_h} \int_T \mathbb{D} \nabla w \cdot \nabla v dT. \tag{6}$$

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Remark

- **Fewest** Degrees of Freedom (DoFs).
- **No** conservation preserving property
- The CG scheme will be **only** used in constructing the preconditioner.

Discontinuous Galerkin FEMs

The interior penalty discontinuous Galerkin (IPDG) numerical algorithm is to find $u_h \in V_{\mathsf{DG}}$ such that

$$A_{\mathsf{DG}}(u_h, v) = \sum_{T\in\mathcal{T}_h} \int_T f v dT, \ \forall v \in V_{\mathsf{DG}}, \text{ where} \tag{7}$$

$$\begin{aligned} A_{\mathsf{DG}}(u_h, v) = & \sum_{T\in\mathcal{T}_h} \int_T \mathbb{D}\nabla u_h \cdot \nabla v dT - \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\mathbb{D}\nabla u_h \cdot \mathbf{n}\}\!\} [\![v]\!] ds \\ - & \ \beta \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\mathbb{D}\nabla v \cdot \mathbf{n}\}\!\} [\![u_h]\!] ds + \sum_{e\in\mathcal{E}_h} \int_e \alpha [\![u_h]\!] [\![v]\!] ds. \end{aligned}$$

Remark:

- $\beta = 1$ - symmetric IPDG scheme; $\beta = -1$ - non-symmetric IPDG scheme
- For $\beta = 1$, α has to be chosen to ensure stability.
- **Features: Flexibility and Conservation**

Choice of Penalty Parameter

We shall only focus on the symmetric IPDG scheme

- Penalty parameter α can be chosen as

$$\alpha_1 = \frac{4k(k+1)D_{\parallel}}{h}, \ \alpha_2 = \frac{4k(k+1)}{h}\mathbf{n}\cdot(\mathbb{D}\mathbf{n}). \tag{8}$$

- Error Estimate

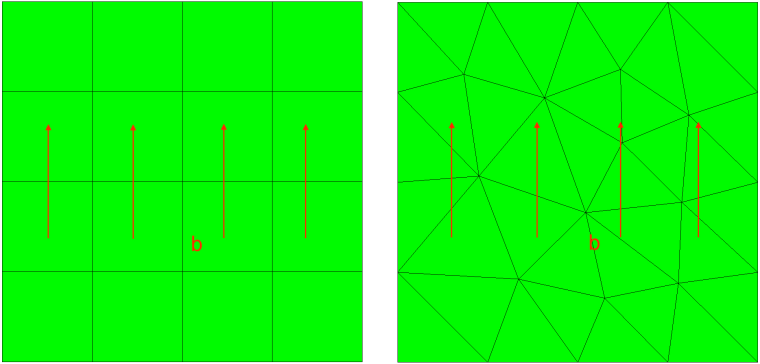
$$\|u-u_h\| \quad \approx \quad \frac{\lambda_{\max}}{\lambda_{\min}} h^{k+1}\|u\|_{k+1} \tag{9}$$

Remark:

- For our case, $\lambda_{\text{min}} = 1 = D_{\perp}$ with varying values in $\lambda_{\text{max}} = D_{\parallel}$.
- The big ratio of anisotropy $\dfrac{\lambda_{\max}}{\lambda_{\min}}$ may destroy the approximation.
- High order scheme with larger k will HELP!
- We may have **bad** condition number.

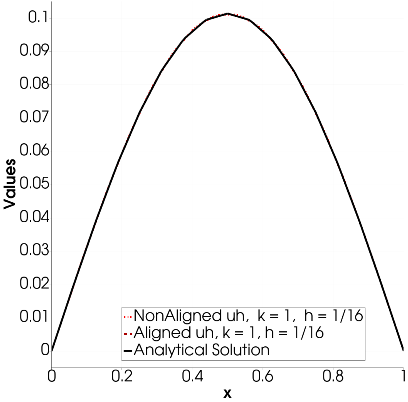
Accuracy Test 1

Set $f = \sin \pi x$ and Dirichelet BC at $x = 1$. Exact solution $u = \frac{1}{\pi^2} \sin \pi x$



(a) Aligned Mesh (b) Non-aligned Mesh

Results on (Non-)Aligned with $h = 1/16$



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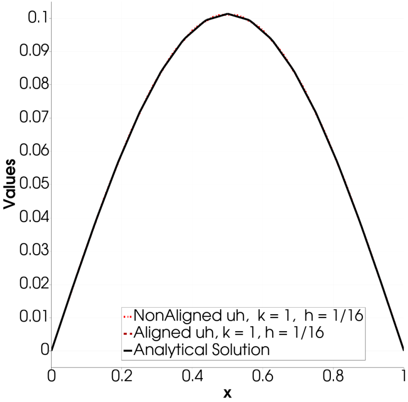
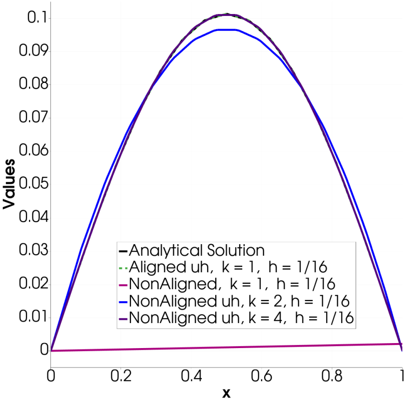


(a) $D_{||} = 1.0$

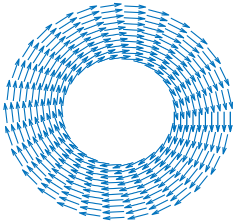


(b) $D_{||} = 10^9$

Accuracy Test 2: Annulus Test

Let Ω be an annulus with R = 1, r = 0.5 and

$$
\begin{aligned}
u &= \sqrt{\frac{3}{4r}}\sin(2\pi r - \pi),\\
b_1 &= \frac{y}{r},\ b_2 = -\frac{x}{r},\ \mathbf{b} = (\mathbf{b_1}, \mathbf{b_2})^\top\\
f &= \sqrt{\frac{3}{4r^5}}(4\pi^2 r^2 - \frac{1}{4})\sin(2\pi r - \pi).
\end{aligned}
$$



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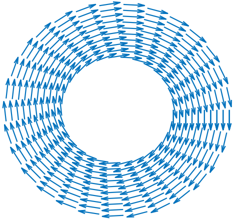
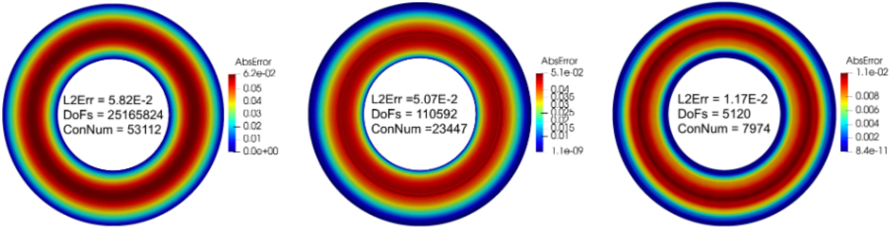


Error Plot $|u - u_h|$ for $D_{||}$ = 1E6 on:
(a) $N_r = 1024, k = 1$; (b) $N_r = 48, k = 2$; (c) $N_r = 8, k = 3$.

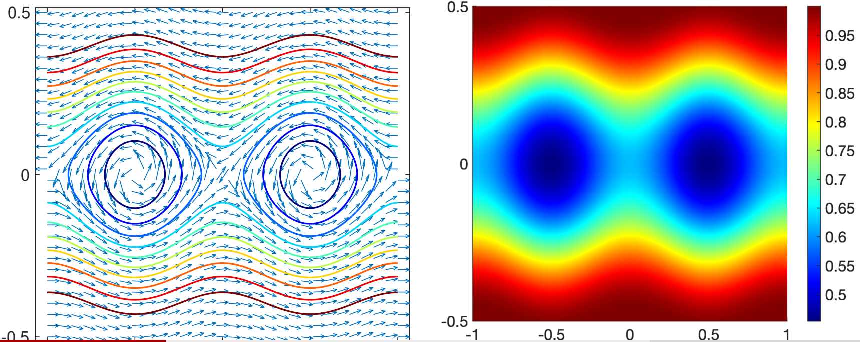


Accuracy Test 3: Two Magnetic Islands

Let $\Omega = [-1,1] \times [-0.5, 0.5]$ and the exact solution be

$$u = \cos\left(\frac{1}{10}\cos(2\pi(x - 3/2)) + \cos(\pi y)\right). \quad (10)$$

$$\mathbf{b} = \frac{\mathbf{B}}{|\mathbf{B}|}, \ \mathbf{B} = \begin{pmatrix} -\pi \sin(\pi y) \\ \frac{2\pi}{10}\sin(2\pi(x - 3/2)) \end{pmatrix}. \quad (11)$$



$1/h$	$D_\parallel = 1E1$ $\|u-u_h\|$	order	$D_\parallel = 1E2$ $\|u-u_h\|$	order	$D_\parallel = 1E4$ $\|u-u_h\|$	order	$D_\parallel = 1E6$ $\|u-u_h\|$	order	$D_\parallel = 1E8$ $\|u-u_h\|$	order
					$k=1$					
8	2.44E-02		5.54E-02		7.12E-02		7.14E-02		7.14E-02	
16	8.31E-03	1.56	3.26E-02	0.77	5.66E-02	0.33	5.70E-02	0.33	5.70E-02	0.33
32	2.39E-03	1.80	1.40E-02	1.22	4.73E-02	0.26	4.85E-02	0.23	4.85E-02	0.23
64	6.33E-04	1.92	4.50E-03	1.64	3.90E-02	0.28	4.27E-02	0.18	4.28E-02	0.18
128	1.62E-04	1.97	1.24E-03	1.86	2.84E-02	0.46	3.86E-02	0.15	3.87E-02	0.14
256	4.10E-05	1.99	3.23E-04	1.94	1.51E-02	0.91	3.54E-02	0.12	3.59E-02	0.11
					$k=2$					
4	5.68E-03		1.20E-02		1.73E-02		1.74E-02		1.74E-02	
8	6.01E-04	3.24	1.36E-03	3.14	7.80E-03	1.15	8.26E-03	1.08	8.26E-03	1.07
16	7.16E-05	3.07	1.22E-04	3.49	3.22E-03	1.28	4.91E-03	0.75	4.94E-03	0.74
32	8.84E-06	3.02	1.11E-05	3.45	4.93E-04	2.71	3.35E-03	0.55	3.56E-03	0.47
64	1.10E-06	3.00	1.19E-06	3.23	3.67E-05	3.75	1.56E-03	1.11	2.85E-03	0.32
128	1.38E-07	3.00	1.40E-07	3.08	2.42E-06	3.92	1.95E-04	3.00	2.23E-03	0.35
					$k=3$					
4	7.04E-04		1.15E-03		6.65E-03		7.34E-03		7.35E-03	
8	4.48E-05	3.98	6.43E-05	4.16	5.26E-04	3.66	9.12E-04	3.01	9.41E-04	2.96
16	2.78E-06	4.01	3.16E-06	4.35	5.17E-05	3.35	1.05E-03	-0.21	1.33E-03	-0.50
32	1.74E-07	4.00	1.80E-07	4.13	1.36E-06	5.25	8.98E-05	3.55	8.84E-04	0.59
64	1.09E-08	4.00	1.10E-08	4.03	3.79E-08	5.16	2.32E-06	5.27	1.32E-04	2.74
128	6.83E-10	4.00	6.83E-10	4.01	1.25E-09	4.92	6.45E-08	5.17	3.37E-06	5.29
					$k=4$					
4	3.64E-05		5.19E-05		9.96E-04		3.08E-03		3.14E-03	
8	1.03E-06	5.15	1.50E-06	5.11	1.70E-05	5.88	9.21E-05	5.06	9.92E-05	4.99
16	3.04E-08	5.08	3.76E-08	5.32	1.34E-07	6.99	1.31E-06	6.14	8.19E-06	3.60
32	9.16E-10	5.05	1.02E-09	5.20	2.45E-09	5.77	9.02E-09	7.18	5.38E-07	3.93
64	2.81E-11	5.03	2.96E-11	5.11	5.56E-11	5.46	4.43E-10	4.35	6.52E-08	3.04
					$k=5$					
2	2.62E-04		3.71E-04		4.12E-03		5.22E-03		5.23E-03	
4	3.12E-06	6.39	4.81E-06	6.27	9.10E-05	5.50	3.43E-04	3.93	3.53E-04	3.89
8	7.58E-08	5.36	8.81E-08	5.77	5.14E-07	7.47	1.59E-05	4.43	9.97E-05	1.82
16	1.09E-09	6.12	1.14E-09	6.27	5.62E-09	6.52	9.92E-08	7.32	2.01E-06	5.63
32	1.66E-11	6.03	1.68E-11	6.09	3.92E-11	7.16	4.08E-10	7.93	1.42E-08	7.15
					$k=6$					
1	1.50E-03		3.18E-03		2.83E-02		5.24E-02		5.29E-02	
2	1.70E-05	6.46	3.72E-05	6.42	7.55E-04	5.23	1.35E-03	5.28	1.36E-03	5.28
4	3.18E-07	5.74	5.40E-07	6.11	5.49E-06	7.10	1.10E-04	3.61	1.73E-04	2.97
8	1.91E-09	7.38	2.67E-09	7.66	2.84E-08	7.60	5.60E-07	7.62	9.63E-06	4.17
16	1.37E-11	7.12	1.63E-11	7.36	7.90E-11	8.49	2.53E-09	7.79	1.57E-08	9.26

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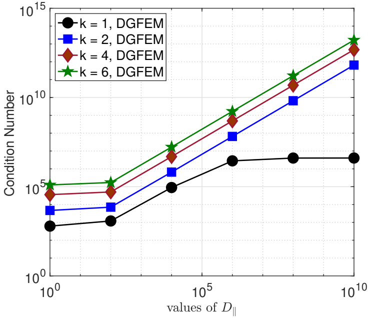

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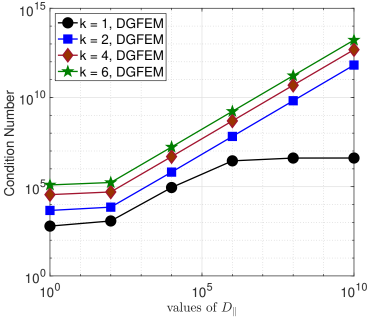

- We need an **effective and efficient fast solver**

Illustration of Auxiliary Space Pre-conditioner (ASP)

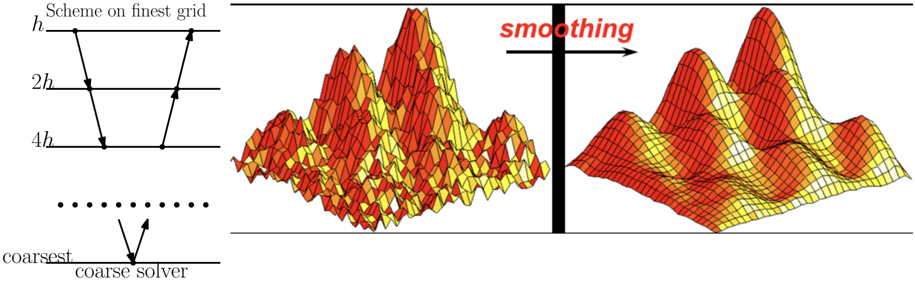

Figure: (a). Multigrid V-cycle;

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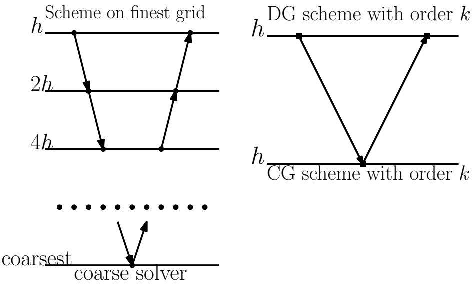

(a) (b)

Figure: (a). Multigrid V-cycle; (b). One-level Auxiliary Space;

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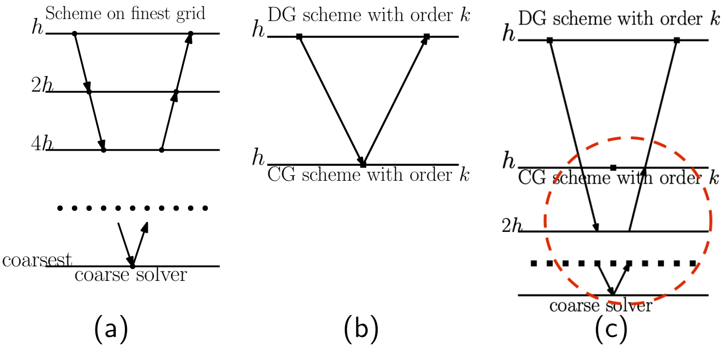

Figure: (a). Multigrid V-cycle; (b). One-level Auxiliary Space; (c). Multi-grid Auxiliary Space.

- Proposed in [S. Nepomnyaschikh 1992]
- b). Use an auxiliary space as "coarse" space, which is easier to solve
- c). Replace the "coarse" solver by existing solvers/preconditioners

Auxiliary Space Pre-conditioner - CG Scheme

Why CG Scheme as the ASP?

- Less DoFs
- Well developed CG - fast solver

Auxiliary Space Pre-conditioner (Efficiency)

Given $\boldsymbol{f} \in V_{\mathsf{DG}}'$, find $\boldsymbol{u} \in V_{\mathsf{DG}}$ such that

$$\boldsymbol{A}_{\mathsf{DG}}\boldsymbol{u} = \boldsymbol{f},$$

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- As in [Antonietti 2017], choose the product of auxiliary spaces: $\bar{V}_{\mathsf{DG}} = V_{\mathsf{DG}} \times V_{\mathsf{CG}}$
- Since $V_{\mathsf{CG}} \subset V_{\mathsf{DG}}$, we can use subset decomposition $V_{\mathsf{DG}} = V_{\mathsf{DG}} + V_{\mathsf{CG}}$.

Auxiliary Space Pre-conditioner (Efficiency)

Given $\boldsymbol{f} \in V'_{\mathsf{DG}}$, find $\boldsymbol{u} \in V_{\mathsf{DG}}$ such that

$$\boldsymbol{A}_{\mathsf{DG}}\boldsymbol{u} = \boldsymbol{f},$$

- As in [Antonietti 2017], choose the product of auxiliary spaces: $\bar{V}_{\mathsf{DG}} = V_{\mathsf{DG}} \times V_{\mathsf{CG}}$
- Since $V_{\mathsf{CG}} \subset V_{\mathsf{DG}}$, we can use subset decomposition $V_{\mathsf{DG}} = V_{\mathsf{DG}} + V_{\mathsf{CG}}$.

Introduce the auxiliary space preconditioner, $\boldsymbol{B}_{\mathsf{DG}} : V'_{\mathsf{DG}} \mapsto V_{\mathsf{DG}}$

$$\boldsymbol{B}_{\mathsf{DG}} = \underbrace{\boldsymbol{S}_{\mathsf{DG}}}_{\text{smoother: Gauss-Seidel}} + \underbrace{\boldsymbol{\Pi}}_{\text{inclusion operator}} \boldsymbol{A}_{\mathsf{CG}}^{-1} \underbrace{\boldsymbol{\Pi}^{\top}}_{L^2 \text{ projection}}, \tag{12}$$

Here $\boldsymbol{\Pi} : V_{\mathsf{CG}} \to V_{\mathsf{DG}}$ and $\boldsymbol{\Pi}^{\top} : V'_{\mathsf{DG}} \to V'_{\mathsf{CG}}$.

Auxiliary Space Pre-conditioner (Efficiency)

Given $\boldsymbol{f} \in V_{\mathsf{DG}}'$, find $\boldsymbol{u} \in V_{\mathsf{DG}}$ such that

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Here $\boldsymbol{\Pi} : V_{\mathsf{CG}} \to V_{\mathsf{DG}}$ and $\boldsymbol{\Pi}^{\top} : V_{\mathsf{DG}}' \to V_{\mathsf{CG}}'$.

Remark:

Here we need to invert $\boldsymbol{A}_{\mathsf{CG}}$ exactly. We shall first show the effectiveness of the proposed pre-conditioner.

Efficiency Test 1 - Solve H^1-problem exactly

- Set $\boldsymbol{B}_{\mathsf{DG}} = \boldsymbol{S}_{\mathsf{DG}} + \boldsymbol{\Pi} \boldsymbol{A}_{\mathsf{CG}}^{-1} \boldsymbol{\Pi}^\top$
- Solve pre-conditioned system by GMRES with tol $=$ 1E-6
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$D_\parallel$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
1	11	11	11	11	11	11	11	11
1E+2	12	10	10	10	10	10	10	11
1E+4	20	8	8	8	8	8	8	9
1E+6	20	14	5	5	5	5	6	7
1E+8	20	4	5	5	5	6	6	5

Table: Iterations for $\boldsymbol{B}_{\mathsf{DG}}$ (solve the H^1 problem exactly) when $h = 1/10$ on non-aligned mesh.

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Remark:

- Almost constant iteration number⟶**Effective and Robust**

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- However, $\boldsymbol{A}_{\text{CG}}^{-1}$ challenging with large size, high polynomial degree, strong anisotropy.

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Table: Iterations for $\boldsymbol{B}_{\mathsf{DG}}$ (solve the H^1 problem exactly) when $h = 1/10$ on non-aligned mesh.

Remark:

• Almost constant iteration number⟶**Effective and Robust**

• However, $\boldsymbol{A}_{\mathsf{CG}}^{-1}$ challenging with large size, high polynomial degree, strong anisotropy. $\rightarrow$ **use preconditioner $B_{\mathsf{CG}} \approx A_{\mathsf{CG}}^{-1}$**

Auxiliary Space Pre-conditioner (Efficiency)

$$\boldsymbol{B}_{\mathsf{DG}} = \boldsymbol{S}_{\mathsf{DG}} + \boldsymbol{\Pi} \boldsymbol{A}_{\mathsf{CG}}^{-1} \boldsymbol{\Pi}^{\top}, \text{ will be replaced by}$$

Auxiliary Space Pre-conditioner (Efficiency)

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\begin{aligned}
\boldsymbol{B}_{\mathsf{DG}} &= \boldsymbol{S}_{\mathsf{DG}} + \boldsymbol{\Pi} \boldsymbol{A}_{\mathsf{CG}}^{-1} \boldsymbol{\Pi}^{\top}, \text{ will be replaced by} \\
\boldsymbol{B}_{\mathsf{DG}}^{\mathsf{inexact}} &= \boldsymbol{S}_{\mathsf{DG}} + \boldsymbol{\Pi} \boldsymbol{B}_{\mathsf{CG}} \boldsymbol{\Pi}^{\top}, \text{ here, } \boldsymbol{B}_{\mathsf{CG}} = \boldsymbol{S}_{\mathsf{CG}} + \boldsymbol{B}_{\mathsf{MG}}
\end{aligned}
$$

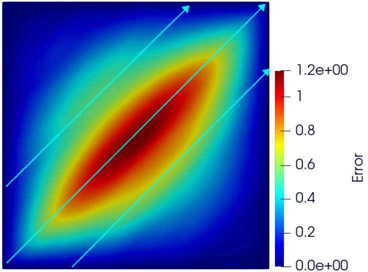
Auxiliary Space Pre-conditioner (Efficiency)

$$\begin{aligned} B_{\mathsf{DG}} &= S_{\mathsf{DG}} + \Pi A_{\mathsf{CG}}^{-1} \Pi^{\top}, \text{ will be replaced by} \\ B_{\mathsf{DG}}^{\mathsf{inexact}} &= S_{\mathsf{DG}} + \Pi B_{\mathsf{CG}} \Pi^{\top}, \text{ here, } B_{\mathsf{CG}} = S_{\mathsf{CG}} + B_{\mathsf{MG}} \end{aligned}$$

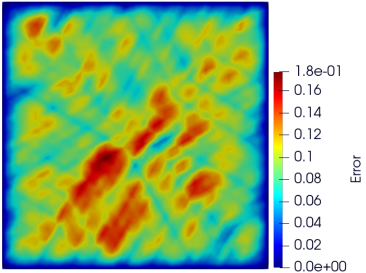
- Here B_{MG}: multi-grid solver for CG-FEM.
- S_{CG}: Schwarz-type block line smoother [Pavarino 1994, Antonietti 2017]

Error plot for $\mathbf{b} = [1,1]^\top/\sqrt{2}$ and exact $u = 0$

Let $\Omega = [-1,1]^2$ and start the iterative solver with a random initial guess.

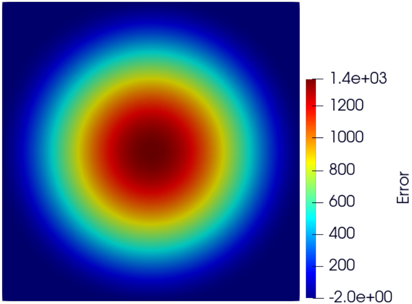


(a) before smoothing

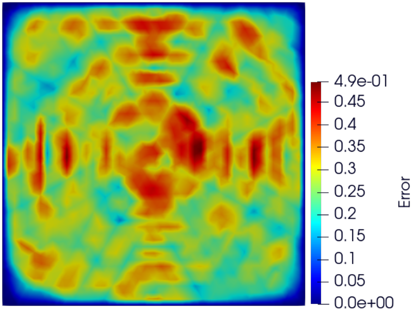


(b) with perpendicular line smoother

Error plot for circular b and exact $u = 0$



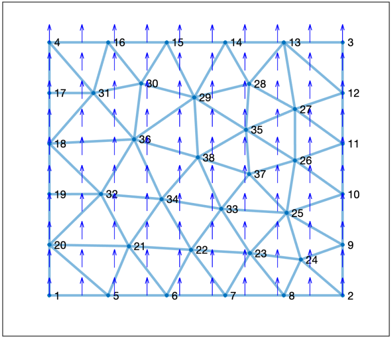
(a) before smoothing



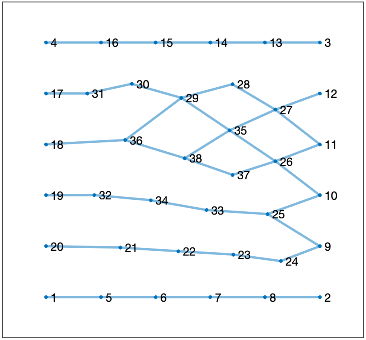
(b) with perpendicular line smoother

- high frequency pattern along the direction perpendicular to the anisotropy

Illustration of Line Smoother for $\mathbf{b} = [0, 1]^\top$

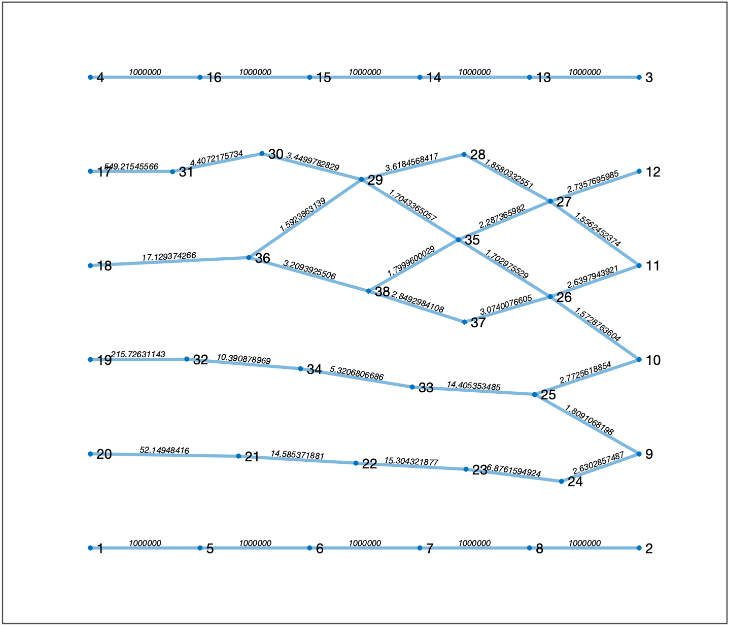


(a) mesh



(b) dropping aligned edges

- Weights: $\omega_e \leftarrow \dfrac{1.0}{|\cos(\theta)| + 10^{-6}}$, $\cos(\theta) \leftarrow \dfrac{\boldsymbol{t}^\top \mathbf{b}_{\text{mid}}}{\|\boldsymbol{t}\| \|\mathbf{b}_{\text{mid}}\|}$
- For example, we choose threshold $\eta = 2.0$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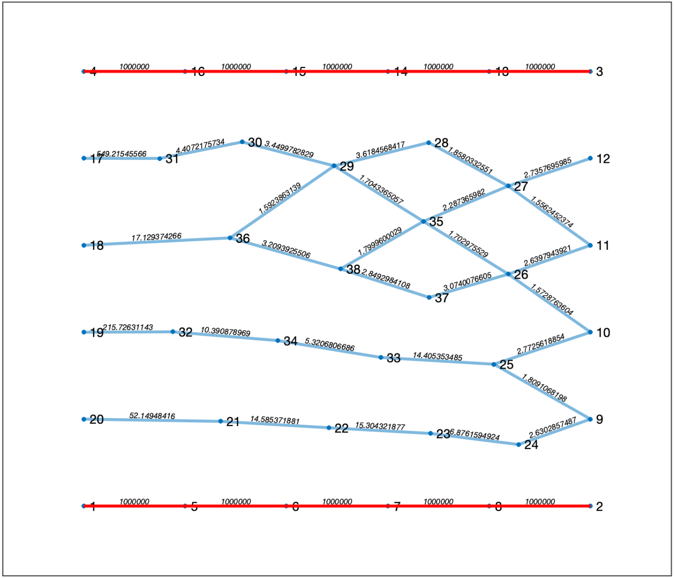
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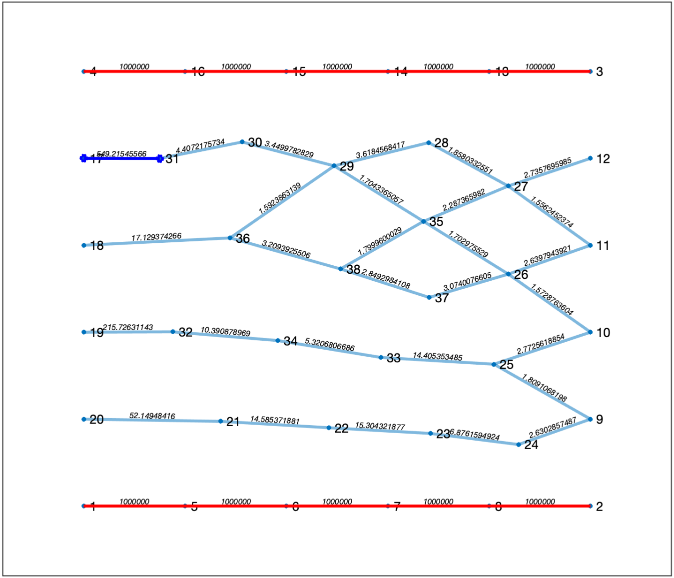
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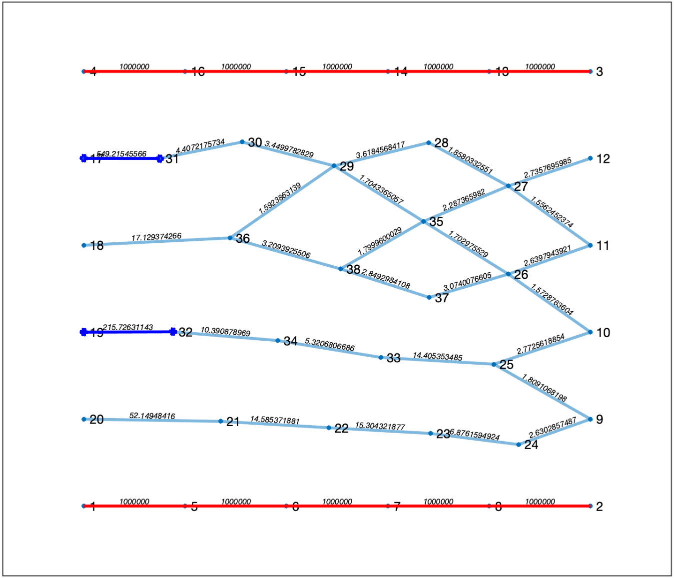
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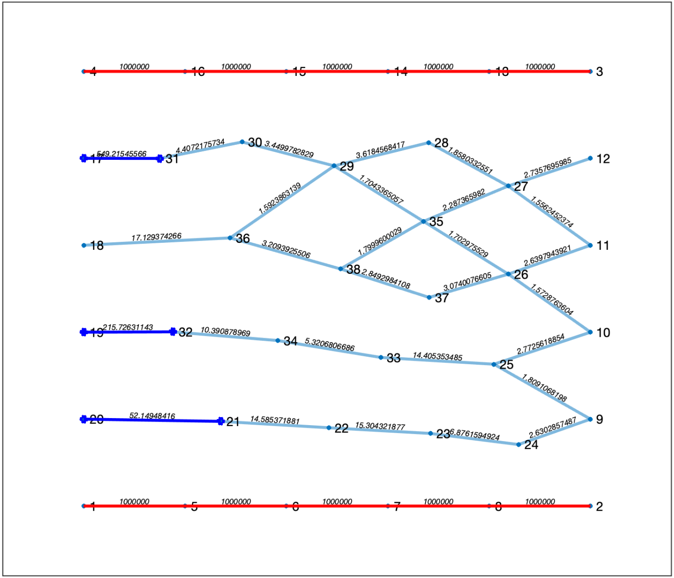
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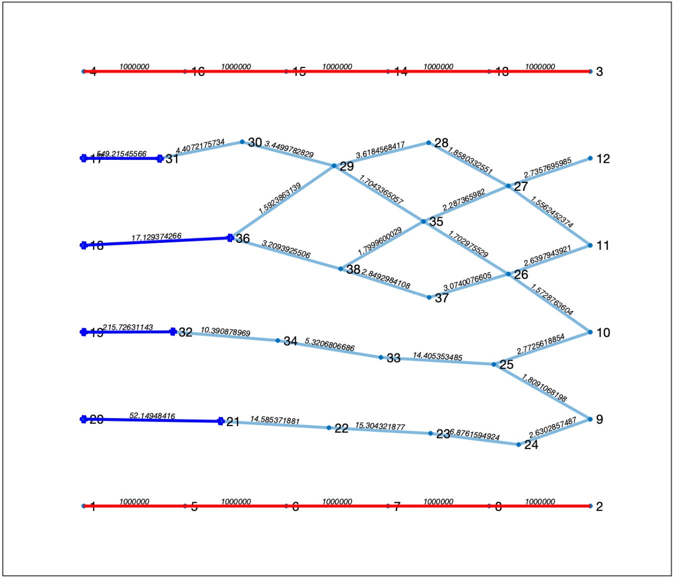
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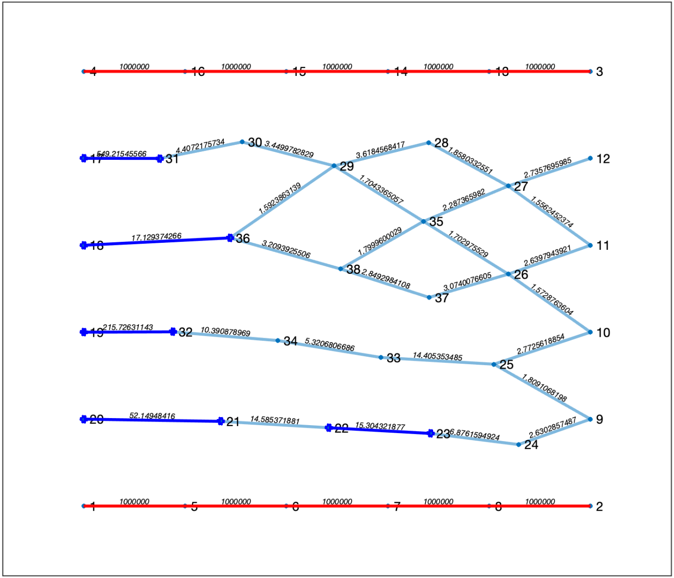
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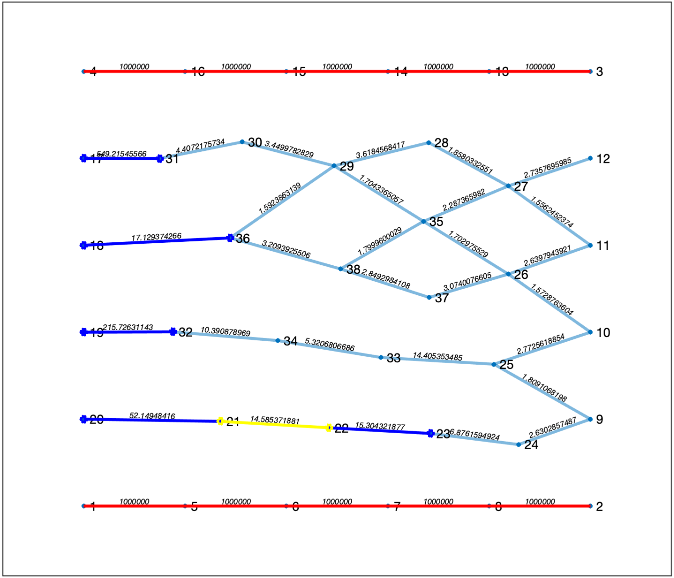
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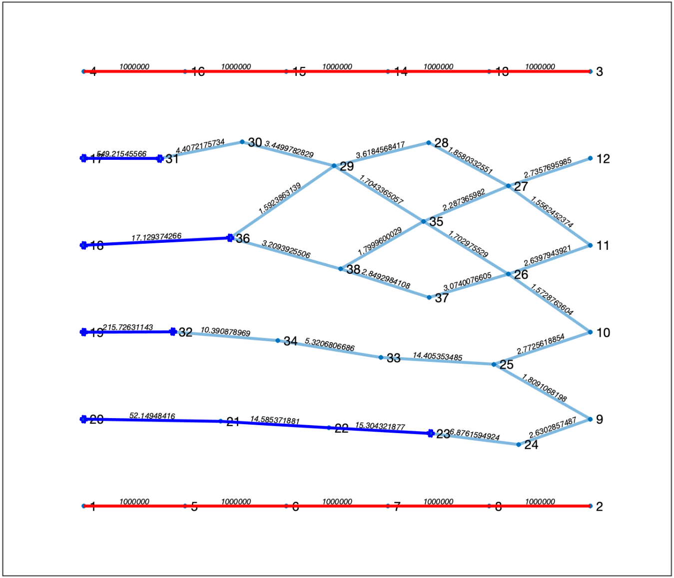

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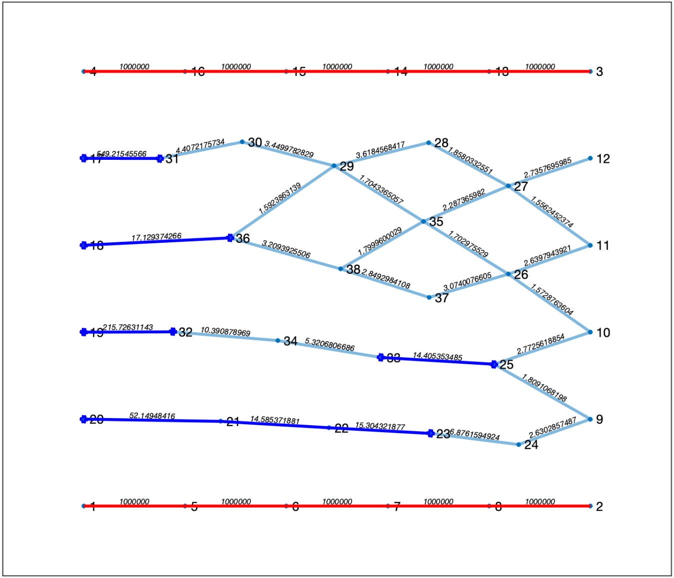
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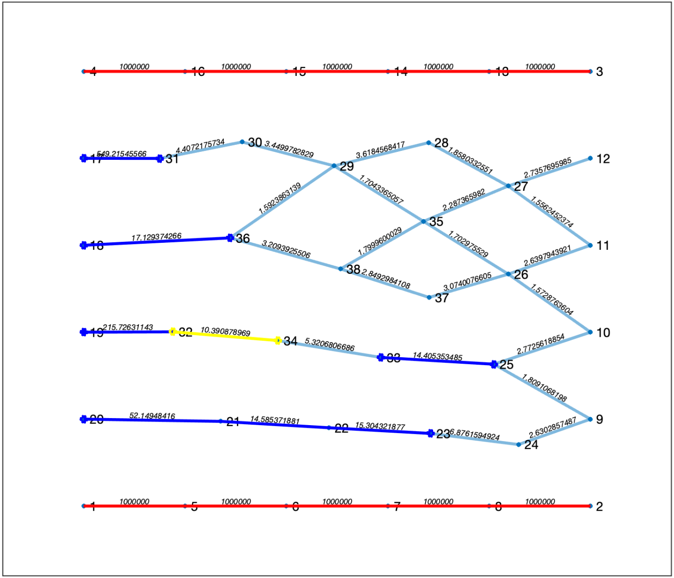
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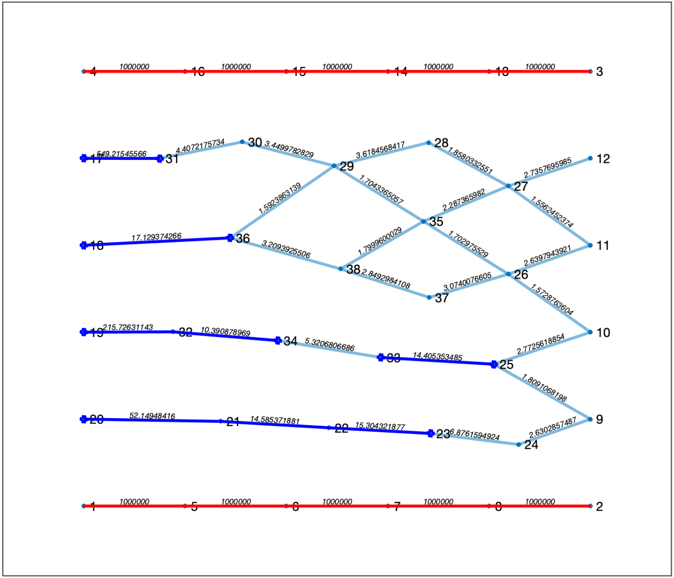
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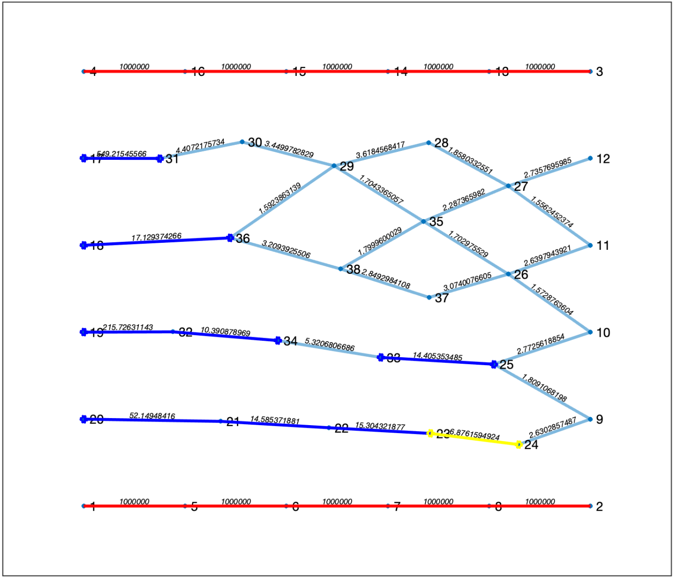
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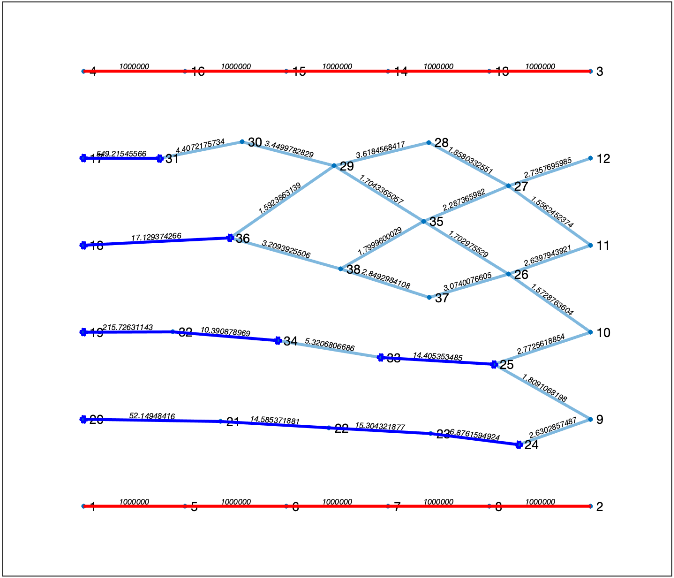
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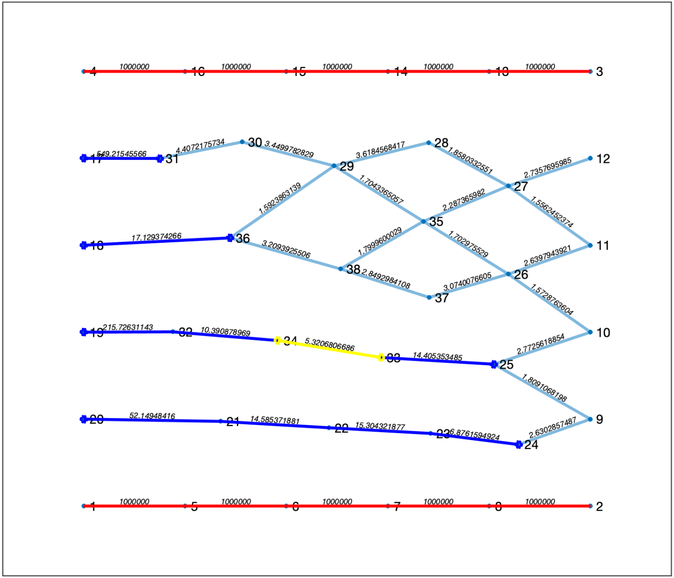
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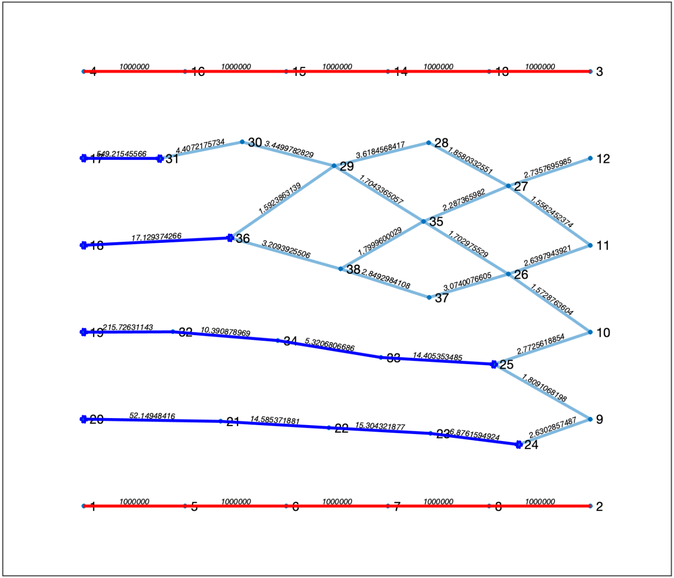
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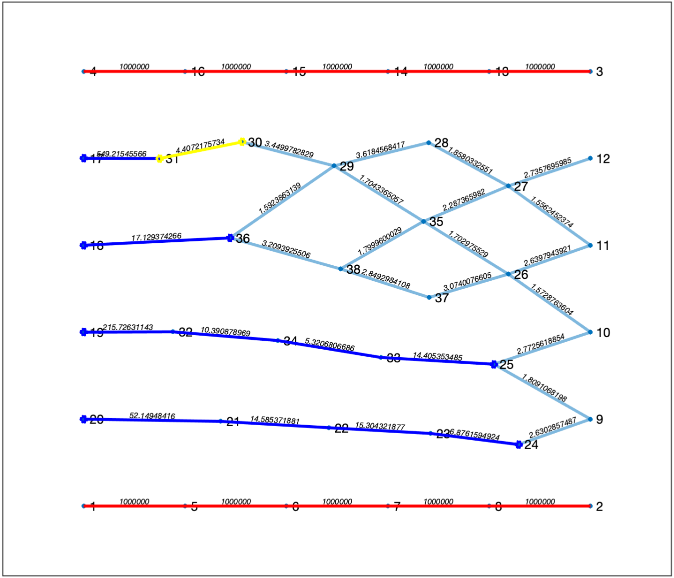
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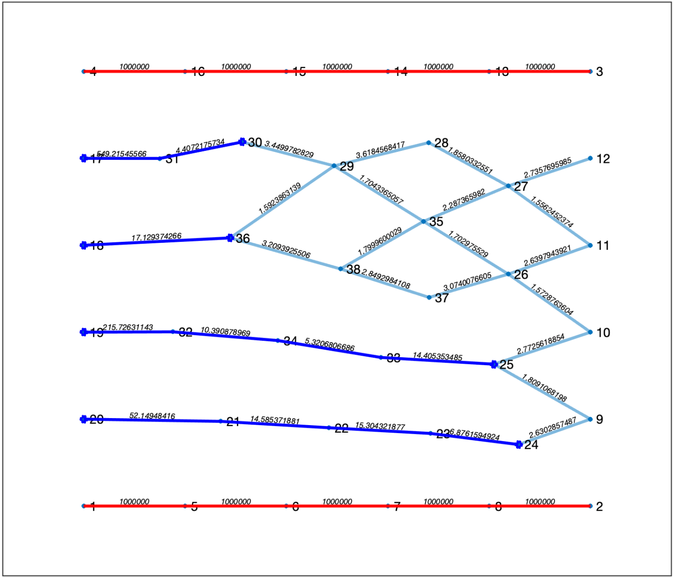
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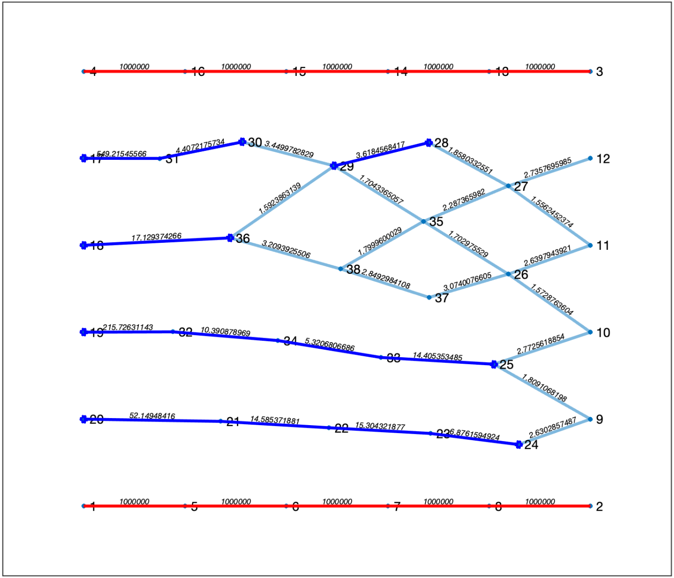

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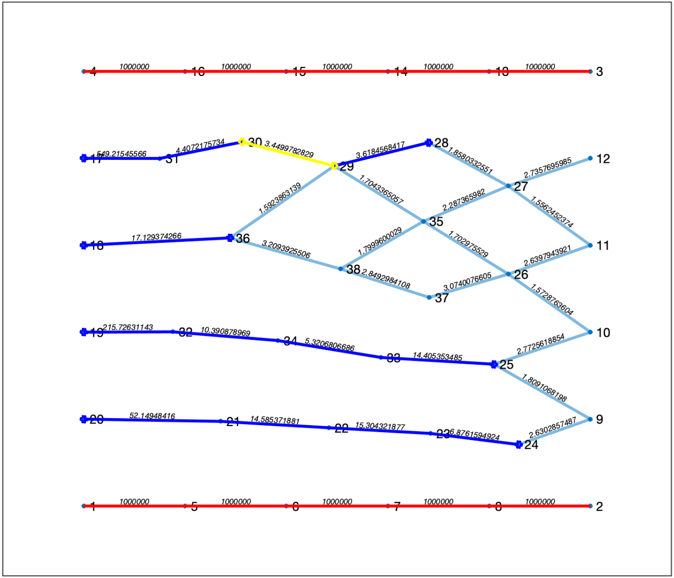

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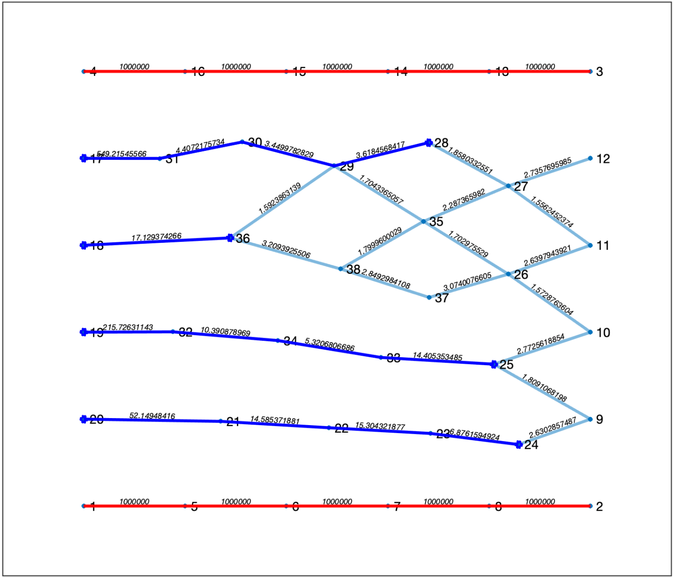
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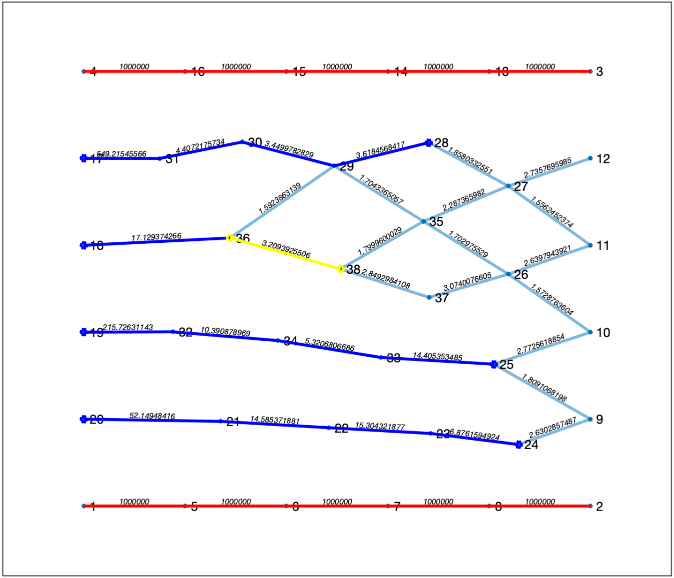
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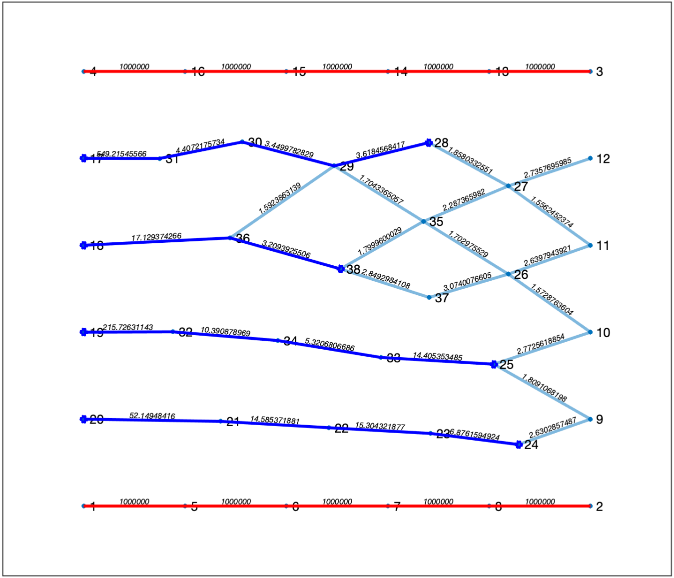
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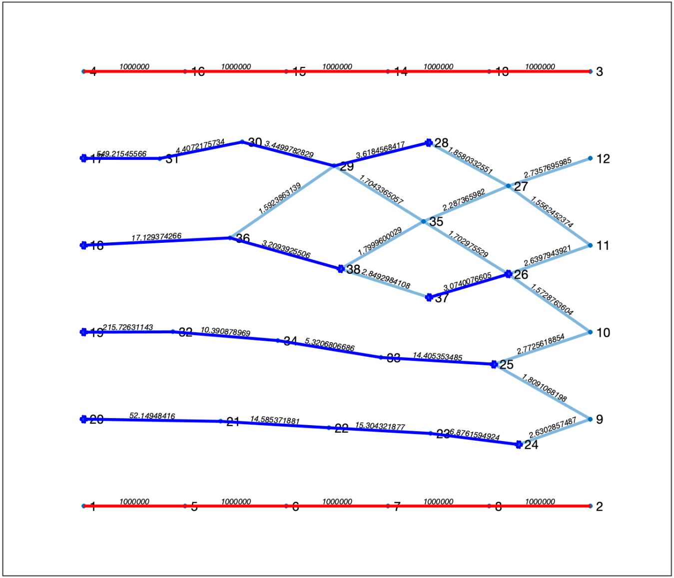
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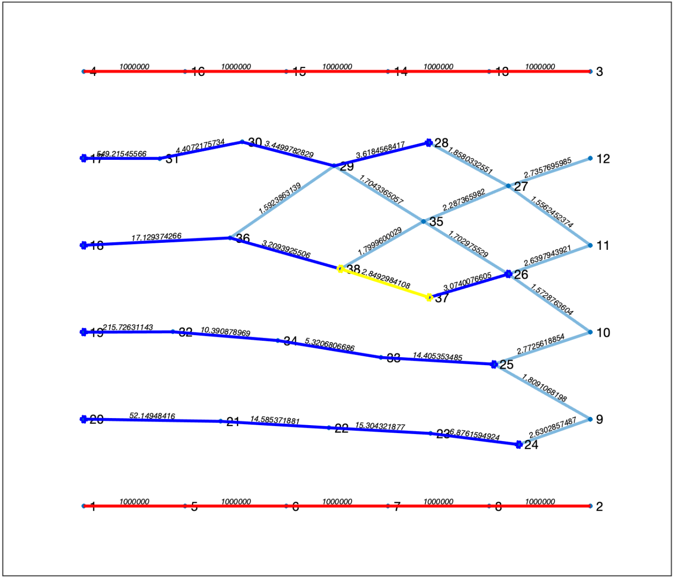
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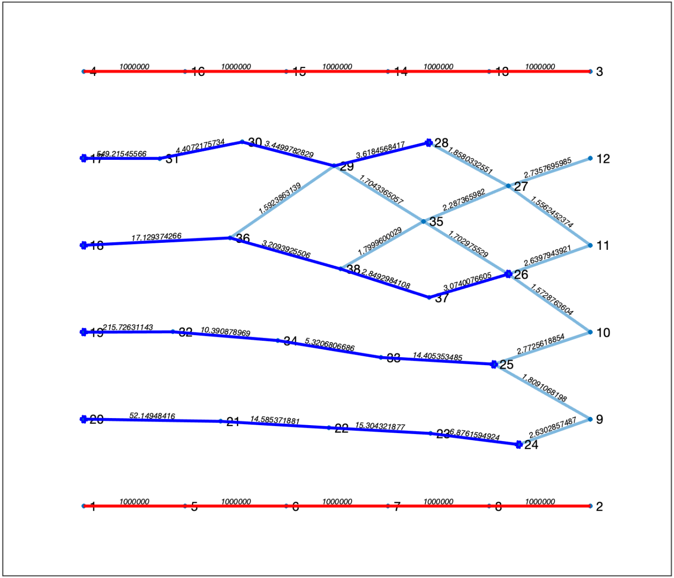

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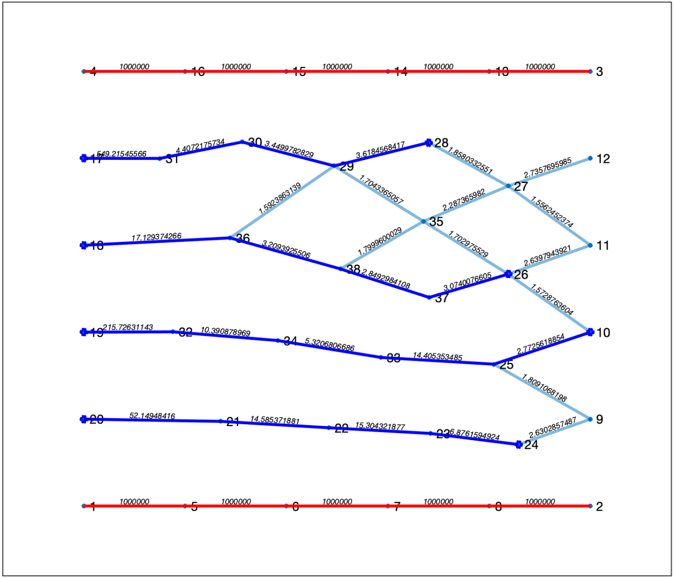

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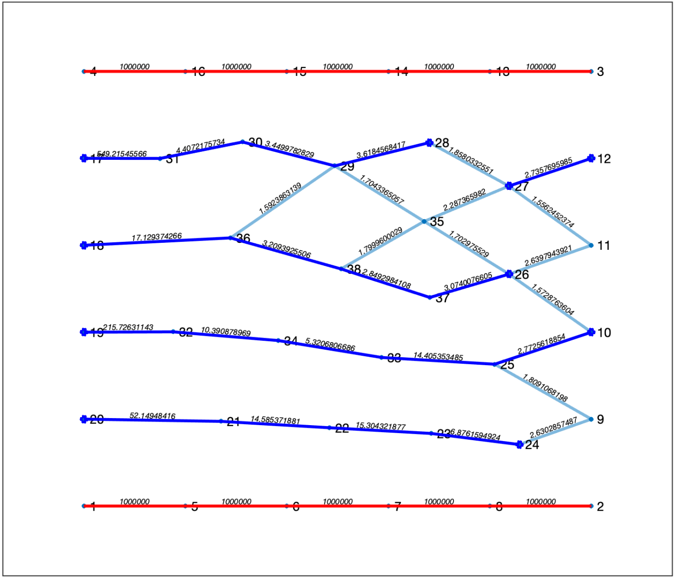
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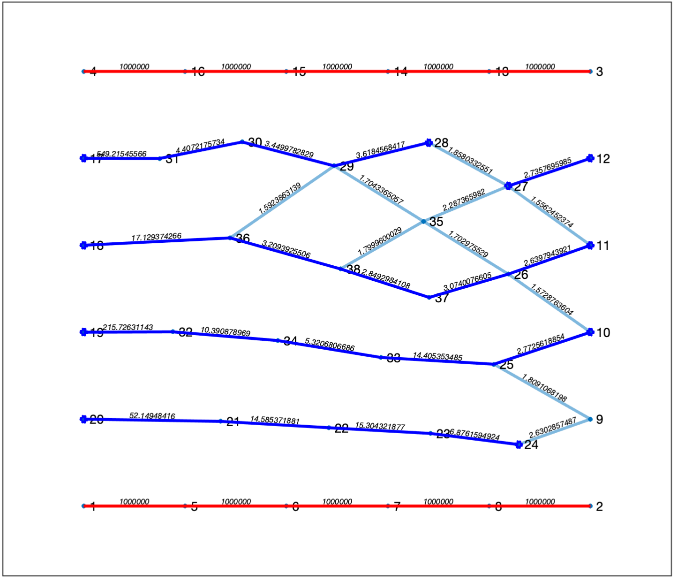
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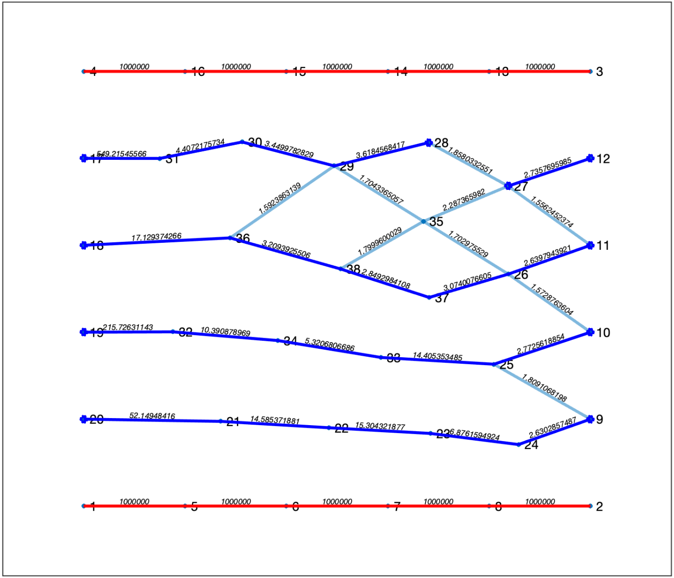
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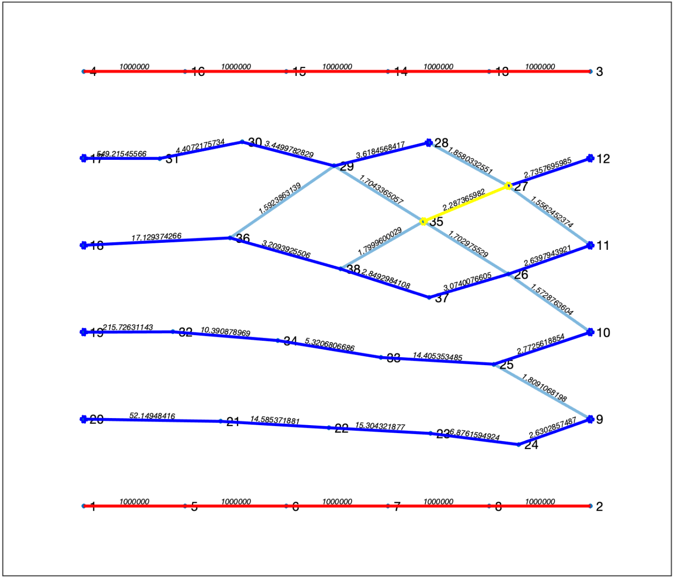
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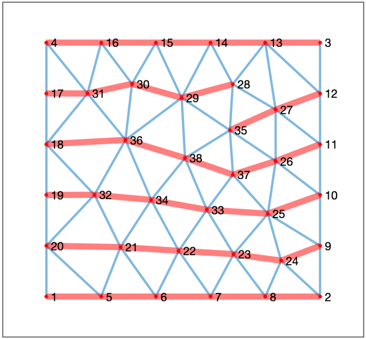

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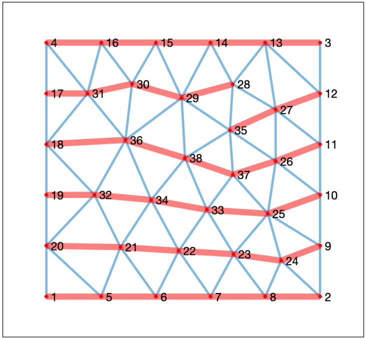
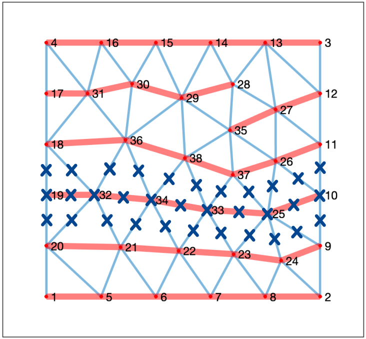


(c) Path Cover



(b) DoFs of one block for P_2 basis

Fast Solver Test 1: $\mathbf{b} = [0, 1]^\top$

- $\mathbf{S}_{\mathsf{DG}}$: symmetric block Gauss-Seidel smoother using vertex patches
- $\mathbf{S}_{\mathsf{CG}}$: block line smoother
- $\mathbf{B}_{\mathsf{MG}}$: NG-Solve build-in MG method for high-order CG
- $\mathbf{A}_{\mathsf{DG}}^{-1} \approx \mathbf{S}_{\mathsf{DG}} + \Pi \mathbf{B}_{\mathsf{CG}} \Pi^\top$ and $\mathbf{B}_{\mathsf{CG}} = \mathbf{S}_{\mathsf{CG}} + \mathbf{B}_{\mathsf{MG}} \approx \mathbf{A}_{\mathsf{CG}}^{-1}$
- $\mathsf{tol}_{\mathsf{DG}} = 10^{-6}$ and $\mathsf{tol}_{\mathsf{CG}} = 10^{-4}$

Fast Solver Test 1: $\mathbf{b} = [0, 1]^\top$

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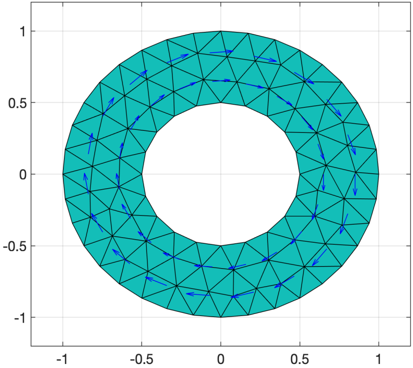
- $\mathbf{B}_{\mathrm{MG}}$: NG-Solve build-in MG method for high-order CG
- $\mathbf{A}_{\mathrm{DG}}^{-1} \approx \mathbf{S}_{\mathrm{DG}} + \Pi \mathbf{B}_{\mathrm{CG}} \Pi^\top$ and $\mathbf{B}_{\mathrm{CG}} = \mathbf{S}_{\mathrm{CG}} + \mathbf{B}_{\mathrm{MG}} \approx \mathbf{A}_{\mathrm{CG}}^{-1}$
- $\mathrm{tol}_{\mathrm{DG}} = 10^{-6}$ and $\mathrm{tol}_{\mathrm{CG}} = 10^{-4}$

$D_{\parallel}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
number of iterations for $B_{\mathrm{DG}}^{\mathrm{inexact}}$ when $h = 1/10$								
1	11	11	11	11	11	11	11	11
1E+2	12	11	10	11	11	11	11	11
1E+4	17	10	9	9	9	9	9	9
1E+6	15	13	10	9	8	7	7	7
1E+8	12	5	6	8	7	7	7	7

Fast Solver Test 2: Circular Test



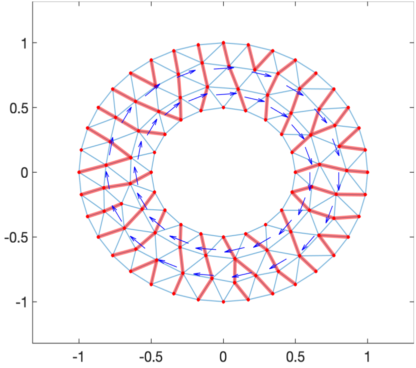


Figure: Illustration of computational Mesh level 1 (left) and line smoother (right).

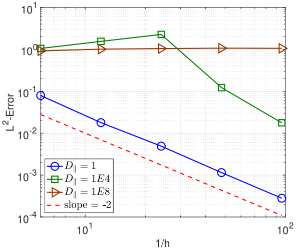
Number of Iterations for Exact Solver

$D_{\|}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
	$h=1/6$							
1	11	10	10	10	10	10	9	9
1E+2	22	15	12	11	11	11	11	11
1E+4	50	52	33	20	13	11	10	10
1E+6	49	61	66	41	39	20	9	9
1E+8	49	66	70	52	39	13	11	11
	$h=1/12$							
1	12	12	12	12	12	12	12	12
1E+2	21	13	11	11	11	11	11	11
1E+4	86	61	27	16	12	10	10	9
1E+6	94	114	91	56	17	8	7	7
1E+8	94	122	125	41	13	11	9	9
	$h=1/24$							
1	13	12	12	12	12	12	12	12
1E+2	21	13	11	11	11	11	11	11
1E+4	111	58	22	14	12	10	10	9
1E+6	205	191	99	52	17	8	7	7
1E+8	210	249	178	41	13	11	9	9

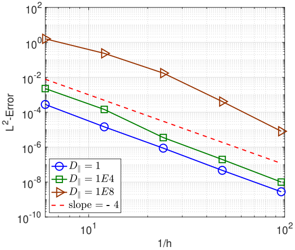
Number of Iterations for In-exact Solver

$D_{\parallel}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
	$h=1/6$							
1	10	9	9	8	8	8	8	8
1E+2	20	13	10	10	10	10	10	10
1E+4	51	52	27	17	11	10	9	9
1E+6	54	65	64	37	24	17	8	16
1E+8	57	73	70	48	10	14	9	10
	$h=1/12$							
1	11	10	10	10	10	10	10	10
1E+2	18	12	10	10	10	10	10	10
1E+4	79	54	20	13	11	9	9	8
1E+6	98	116	84	32	19	16	8	9
1E+8	109	129	85	44	16	12	9	9
	$h=1/24$							
1	13	12	12	12	12	12	12	12
1E+2	21	13	11	11	11	11	11	11
1E+4	111	58	24	19	16	13	11	10
1E+6	205	191	122	87	19	16	8	9
1E+8	210	249	178	44	15	12	9	9

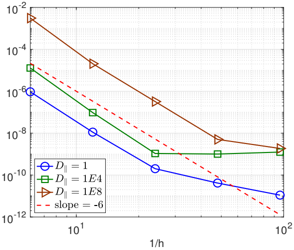
Accuracy Test for Annulus Test with Varying Values in $D_\parallel$



(a). $k = 1$



(b). $k = 3$



(c). $k = 5$

Fast Solver Test 3: WEST tokamak 2D with circular $\mathbf{b}$

A Gaussian source is diffused towards an identical sink

$$f_{sc} = D_{\|} \exp\left(-r_{sc}^2/0.05^2\right)$$
$$f_{sk} = -D_{\|} \exp\left(-r_{sk}^2/0.05^2\right),$$

where $r_{sc} = \sqrt{(x-1.5)^2 + y^2}$ and $r_{sk} = \sqrt{(x+1.5)^2 + y^2}$. The magnetic field direction is chosen as $\mathbf{b} = (b_1, b_2)^\top$ where

$$b_1 = \frac{y}{r}, \; b_2 = -\frac{x}{r}.$$

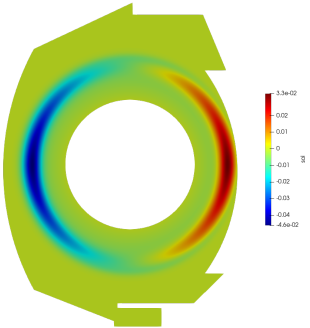
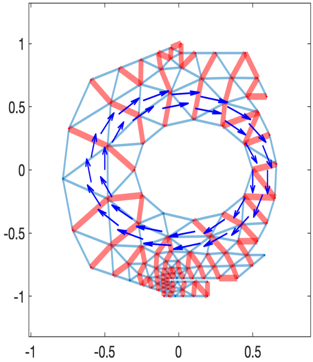
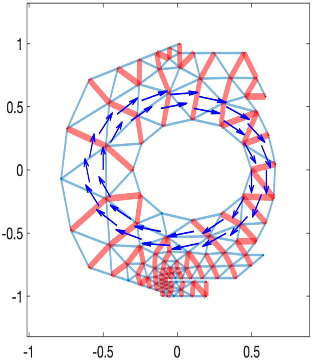


Figure: Plot of solution with $D_{\|}$ =1E4.

Illustration of line smoother



(a) η= 1.05



(b) η=1.2

Numerical Performance for Solvers

$D_{\parallel}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
number of iterations for $\boldsymbol{B}_{\mathsf{DG}}$ when $h=1/10$								
1	8	8	8	8	8	8	7	7
1E+2	17	14	12	11	11	11	11	11
1E+4	21	20	18	15	15	14	13	13
1E+6	21	20	18	17	17	16	17	18
1E+8	21	20	18	17	17	16	15	18
number of iterations for $\boldsymbol{B}_{\mathsf{DG}}^{\mathsf{inexact}}$ when $h=1/10$								
1	8	8	8	8	8	8	7	7
1E+2	17	13	11	11	10	10	10	10
1E+4	27	26	20	18	15	13	12	12
1E+6	28	30	26	24	23	22	21	20
1E+8	28	30	29	28	26	26	25	25

Conclusions and Future Work

Conclusion

- High order scheme can resolve the numerical pollution on the non-aligned mesh
- Auxiliary Space Preconditioner (ASP) is efficient and effective in solving the linear system with large anisotropy

Future Work

- Numerical analysis
- Incorporate with M. Stowell and D. Copeland in MFEM implementation
- More applications with complicated magnetic fields

Thank you!

Reference

- D. Green, X. Hu, J. Lore, L. Mu, and M. Stowell, An Efficient High-order Numerical Solver for Diffusion Equations with Strong Anisotropy, CPC, 276, 2022.
- D. Green, X. Hu, J. Lore, L. Mu, and M. Stowell, An Efficient High-order Solver for Diffusion Equations with Strong Anisotropy on Non-anisotropy-aligned Meshes, Submitted.