High-Order Solvers in MFEM

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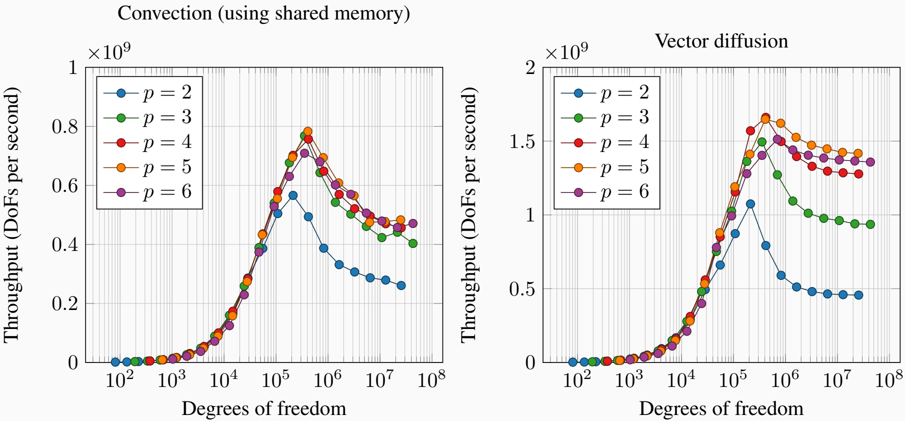
High-Order Discretizations and Solvers

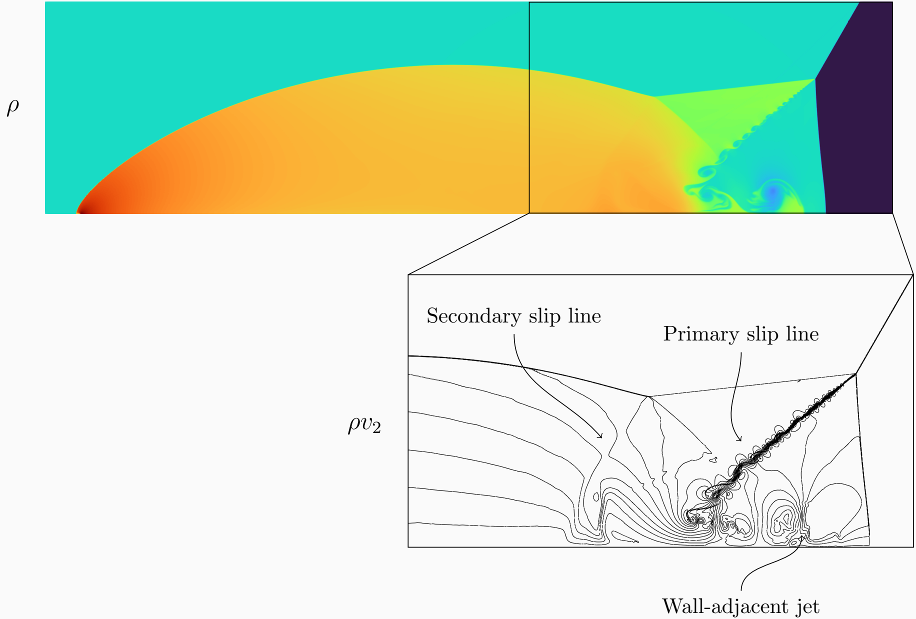
- High-order discretizations present many benefits
  - Very high accuracy on smooth problems
  - With hp-refinement, exponential convergence on problems with singularities
  - Advection-dominated problems: low dispersion, dissipation, better resolution of small-scale features
  - High arithmetic intensity $\implies$ better for GPUs

For example...

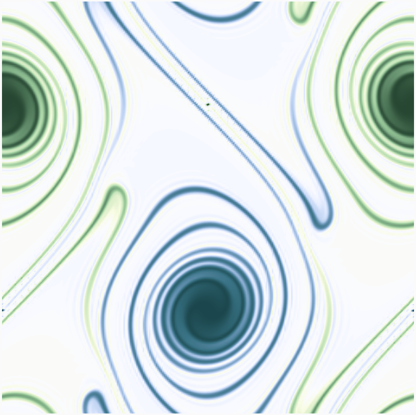
Performant GPU Kernels

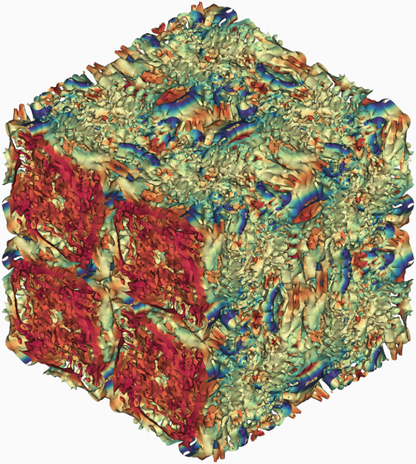


Double Mach Reflection

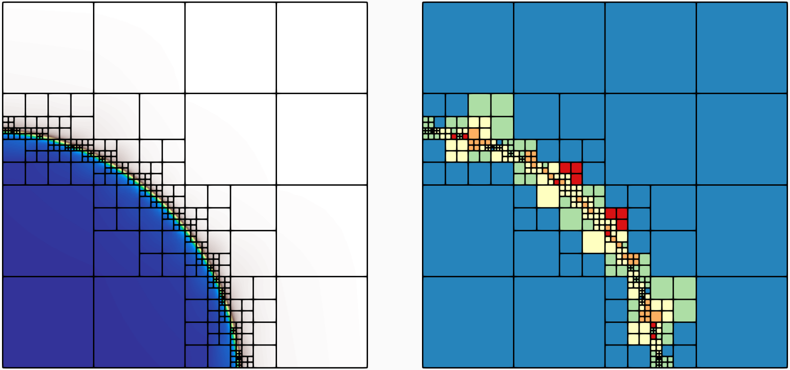


Incompressible Flow





hp-Adaptivity



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- Difficult to solve the linear systems
  - Poorly conditioned systems ($\kappa \sim p^3/h^2$ or $\kappa \sim p^4/h^2$)
  - Large, dense, element-wise blocks ($\mathcal{O}(p^6)$ nonzeros in 3D)
  - **Expensive** to form/assemble the matrix
  - Classical methods don't always work well

High-Order Solvers in MFEM

MFEM aims to make **efficient solvers** for **high-order** problems **readily available** and **easy to use**

MFEM's Solver Capabilities

Traditional Solver Methods

Iterative methods

- CG, GMRES, FGMRES, MINRES, BiCGSTAB

Sparse direct methods

- UMFPACK, KLU, PARDISO, SuperLU, MUMPS, STRUMPACK

Multigrid and AMG methods

- *hypre*, AMGX, Ginkgo, geometric multigrid

Solvers for High-Order

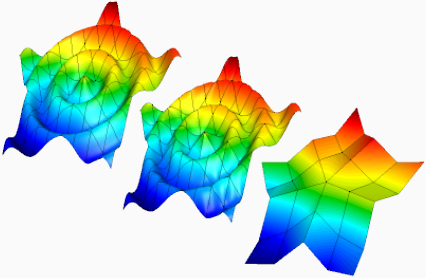
MFEM also provides several specialized high-order solvers

Solvers Designed for High-Order Methods

Solver	Model Problem	Matrix-Free	GPU
hp-multigrid	Diffusion + others...	✓	✓
Low-order refined	Diffusion, curl-curl, grad-div	✓	✓
Matrix-free AMS	curl-curl	✓	✓
Block ILU	Advection-dominated	✗	✗
AIR	Advection-dominated	✗	✗

Geometric multigrid

- Flexible base class `Multigrid`
- BYO:
  - Hierarchy of operators
  - Prolongation operators
  - Smoothers
  - Coarse solvers



- `GeometricMultigrid`
  - Work on a hierarchy of 'FiniteElementSpace' objects
  - h and p refinements supported
- Illustrated in `ex26` and `ex26p`

Matrix-Free Multigrid

- `FiniteElementSpaceHierarchy` (as the name suggests...) manages a hierarchy of finite element spaces
- The user can add h- or p-refined spaces
- Prolongation operators between p-refined spaces are automatically **matrix-free** and **sum-factorized**
- `OperatorJacobiSmoother` and `OperatorChebyshevSmoother` provide **fast**, **matrix-free**, **sum-factorized** access to the diagonal of the high-order operator and associated Chebyshev smoother

Demo

Low-order refined

Main idea

- High-order matrix-free operator — **no assembled matrix**!
- Assemble low-order matrix on auxiliary refined mesh

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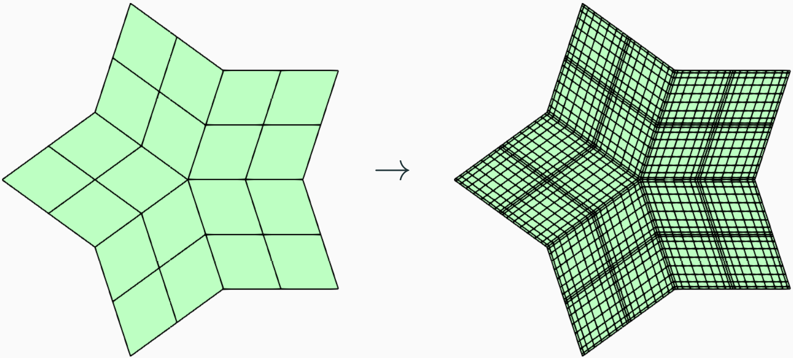

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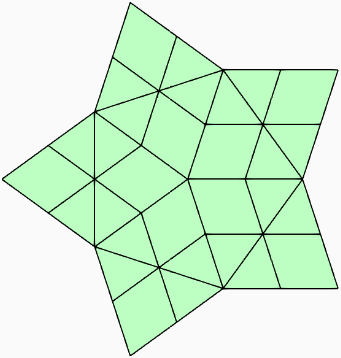

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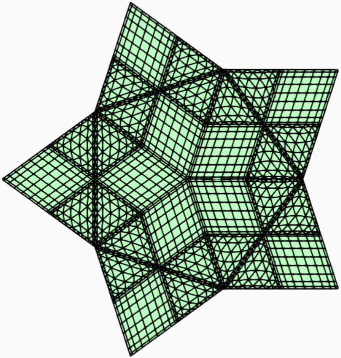

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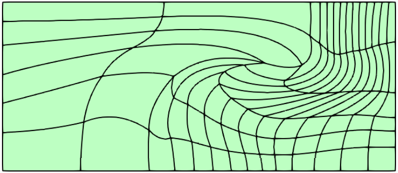

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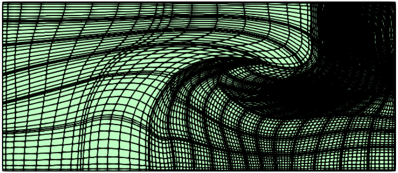

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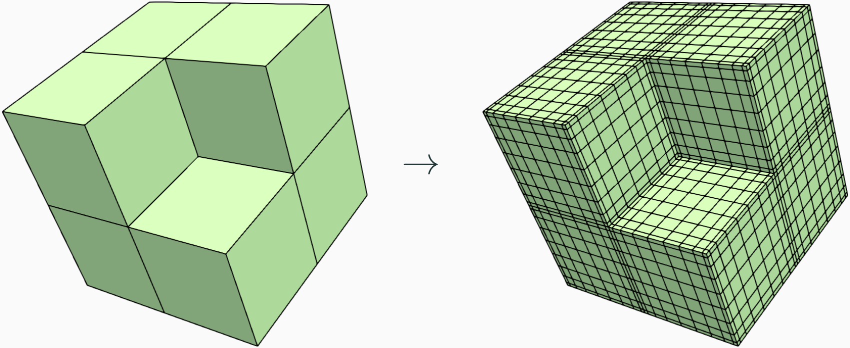

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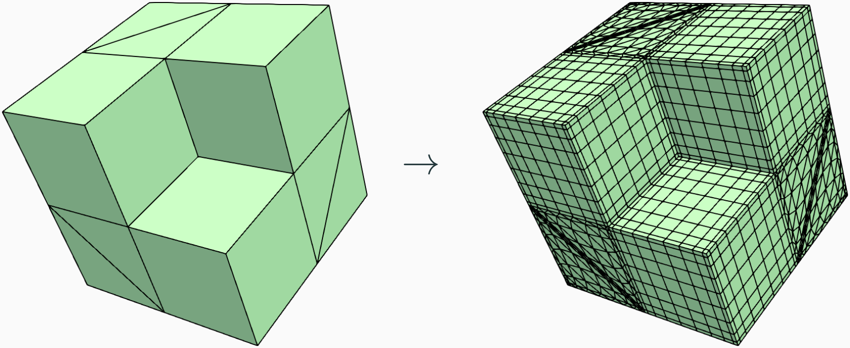

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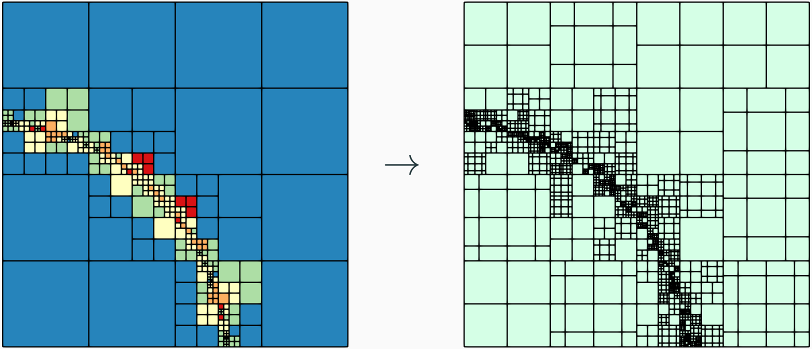

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    - H^1 diffusion
    - Definite Maxwell with $H(\mathrm{curl})$
    - grad-div problem with $H(\mathrm{div})$
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- Parallel example `plor_solvers` provides matrix-free AMG solvers for these problems
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- One-liners:
  - `LORSolver<UMFPackSolver> solv_lor(a, ess_dofs);`
  - `LORSolver<HypreBoomerAMG> solv_lor(a, ess_dofs);`

Demo

Advection-dominated problems

- More challenging (from a methods point of view) than elliptic/parabolic problems
- Most robust options are currently **matrix-based**
    - Block ILU(0) with MDF ordering
    - Approximate Ideal Restriction (AIR) AMG
- Both illustrated in `ex9p`
- Both methods can perform “sweeps”, i.e. converge immediately if there is a perfect (triangular) ordering (no cycles)

Demo

- Questions?
- Feedback?
- Suggestions?