

High-Order Solvers in MFEM

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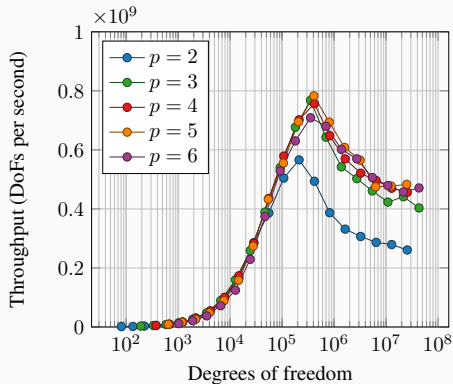
High-Order Discretizations and Solvers

- High-order discretizations present many benefits
 - Very high accuracy on smooth problems
 - With hp -refinement, exponential convergence on problems with singularities
 - Advection-dominated problems: low dispersion, dissipation, better resolution of small-scale features
 - High arithmetic intensity \implies better for GPUs

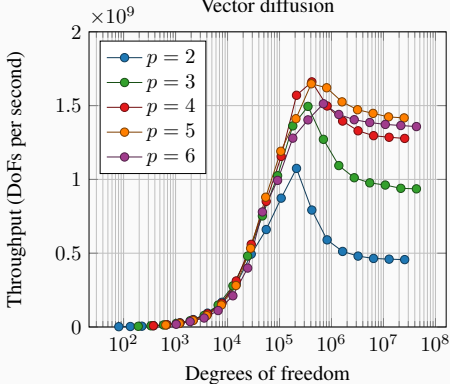
For example...

Performant GPU Kernels

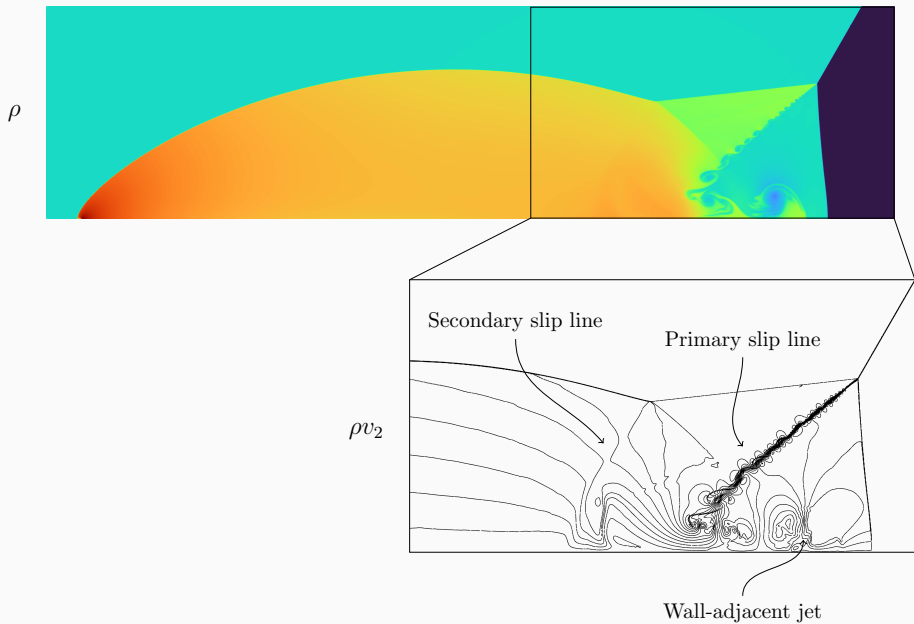
Convection (using shared memory)



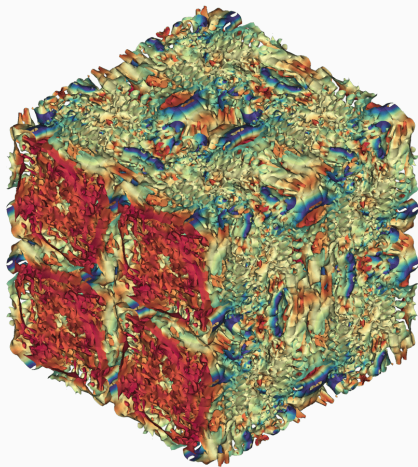
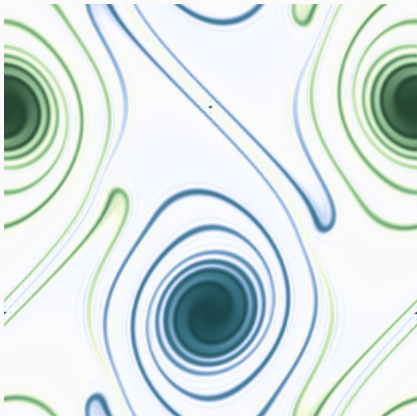
Vector diffusion



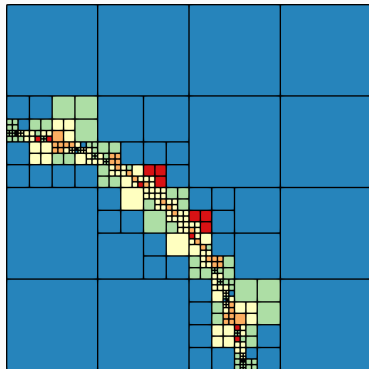
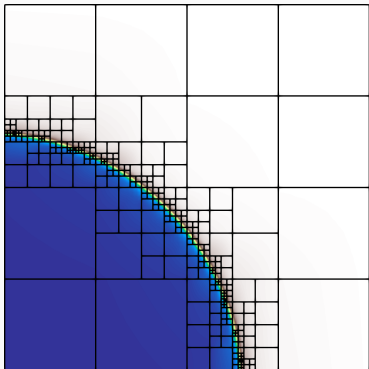
Double Mach Reflection



Incompressible Flow



hp -Adaptivity



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But...

- Difficult to solve the linear systems
 - Poorly conditioned systems ($\kappa \sim p^3/h^2$ or $\kappa \sim p^4/h^2$)
 - Large, dense, element-wise blocks ($\mathcal{O}(p^6)$ nonzeros in 3D)
 - **Expensive** to form/assemble the matrix
 - Classical methods don't always work well

High-Order Solvers in MFEM

MFEM aims to make **efficient solvers** for **high-order** problems **readily available** and **easy to use**

MFEM's Solver Capabilities

Traditional Solver Methods

Iterative methods

- CG, GMRES, FGMRES, MINRES, BiCGSTAB

Sparse direct methods

- UMFPACK, KLU, PARDISO, SuperLU, MUMPS, STRUMPACK

Multigrid and AMG methods

- *hypra*, AMGX, Ginkgo, geometric multigrid

Solvers for High-Order

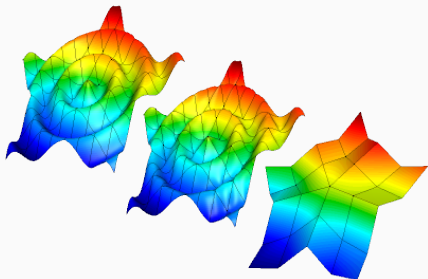
MFEM also provides several specialized high-order solvers

Solvers Designed for High-Order Methods

Solver	Model Problem	Matrix-Free	GPU
<i>hp</i> -multigrid	Diffusion + others...	✓	✓
Low-order refined	Diffusion, curl-curl, grad-div	✓	✓
Matrix-free AMS	curl-curl	✓	✓
Block ILU	Advection-dominated	✗	✗
AIR	Advection-dominated	✗	✗

Geometric multigrid

- Flexible base class `Multigrid`
- BYO:
 - Hierarchy of operators
 - Prolongation operators
 - Smoothers
 - Coarse solvers
- `GeometricMultigrid`
 - Work on a hierarchy of 'FiniteElementSpace' objects
 - h and p refinements supported
- Illustrated in `ex26` and `ex26p`



Matrix-Free Multigrid

- `FiniteElementSpaceHierarchy` (as the name suggests...) manages a hierarchy of finite element spaces
- The user can add h - or p -refined spaces
- Prolongation operators between p -refined spaces are automatically **matrix-free** and **sum-factorized**
- `OperatorJacobiSmoother` and `OperatorChebyshevSmoother` provide **fast, matrix-free, sum-factorized** access to the diagonal of the high-order operator and associated Chebyshev smoother

Demo

Low-order refined

Main idea

- High-order matrix-free operator — **no assembled matrix!**
- Assemble low-order matrix on auxiliary refined mesh

Low-order refined

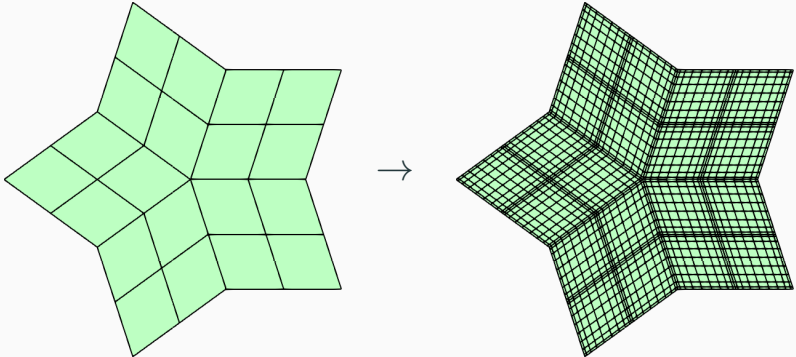
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 - Under some assumptions, the two are **spectrally equivalent**
 - Use your favorite classical preconditioner on the LOR system

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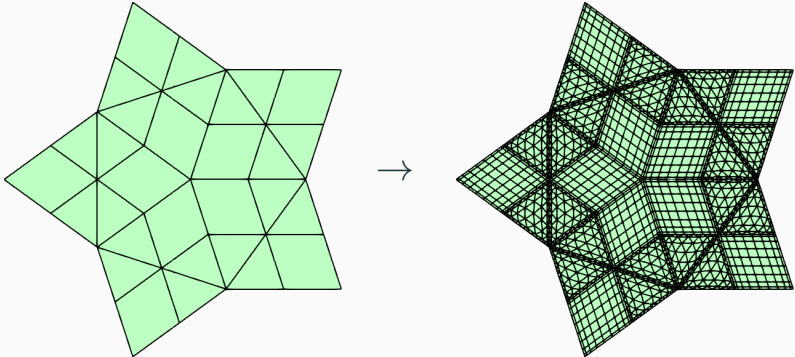
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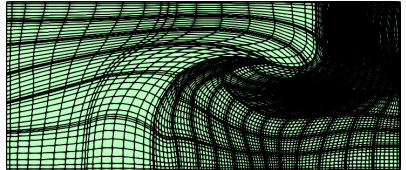
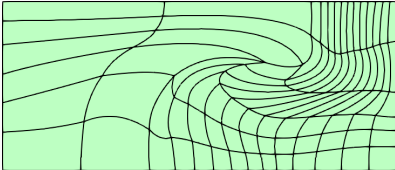
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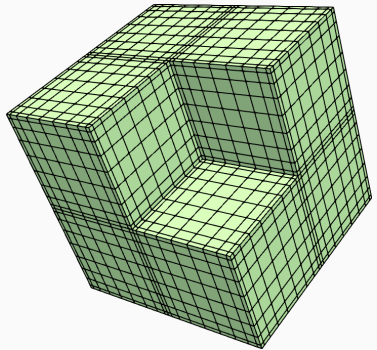
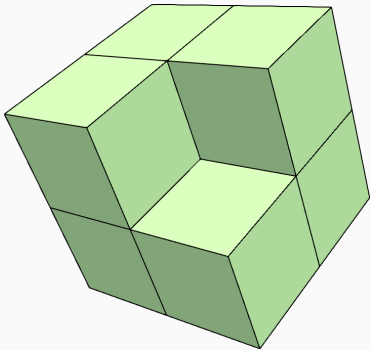
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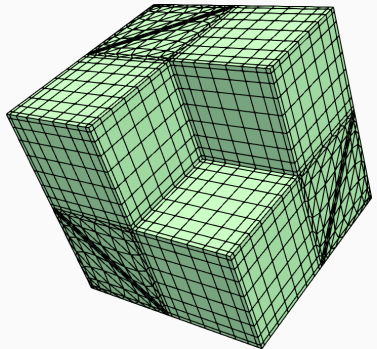
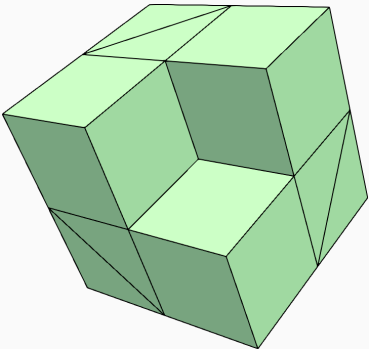
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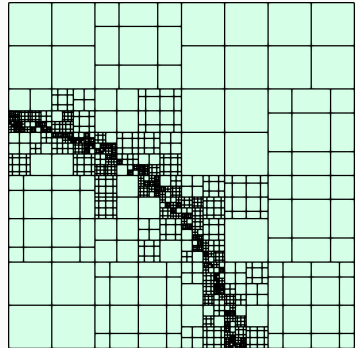
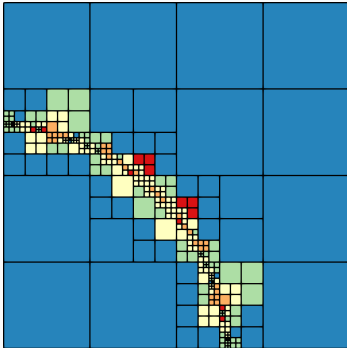
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 - Definite Maxwell with $H(\text{curl})$
 - grad-div problem with $H(\text{div})$
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- One-liners:
 - `LORSolver<UMFPackSolver> solv_lor(a, ess_dofs);`
 - `LORSolver<HypreBoomerAMG> solv_lor(a, ess_dofs);`

Demo

Advection-dominated problems

- More challenging (from a methods point of view) than elliptic/parabolic problems
- Most robust options are currently **matrix-based**
 - Block ILU(0) with MDF ordering
 - Approximate Ideal Restriction (AIR) AMG
- Both illustrated in `ex9p`
- Both methods can perform “sweeps”, i.e. converge immediately if there is a perfect (triangular) ordering (no cycles)

Demo

- Questions?
- Feedback?
- Suggestions?

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